

ELLIPTIC EQUATIONS WITH NONLINEAR ABSORPTION DEPENDING ON THE SOLUTION AND ITS GRADIENT

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ABSTRACT. We study positive solutions of equation (E1) $-\Delta u + u^p |\nabla u|^q = 0$ ($0 \leq p, 0 \leq q \leq 2, p+q > 1$) and (E2) $-\Delta u + u^p + |\nabla u|^q = 0$ ($p > 1, 1 < q \leq 2$) in a smooth bounded domain $\Omega \subset \mathbb{R}^N$. We obtain a sharp condition on p and q under which, for every positive, finite Borel measure μ on $\partial\Omega$, there exists a solution such that $u = \mu$ on $\partial\Omega$. Furthermore, if the condition mentioned above fails then any isolated point singularity on $\partial\Omega$ is removable, namely there is no positive solution that vanishes on $\partial\Omega$ everywhere except at one point. With respect to (E2) we also prove uniqueness and discuss solutions that blow-up on a compact subset of $\partial\Omega$. In both cases we obtain a classification of positive solutions with an isolated boundary singularity. Finally, in Appendix A a uniqueness result for a class of quasilinear equations is provided. This class includes (E1) when $p = 0$ but not the general case.

Keywords: quasilinear equations, boundary singularities, Radon measures, Borel measures, weak singularities, strong singularities, boundary trace, removability.

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1. INTRODUCTION

In this paper, we are concerned with the boundary value problems with measure data for equations of the form

$$(1.1) \quad -\Delta u + H(x, u, \nabla u) = 0$$

in Ω where Ω is a C^2 bounded domain in \mathbb{R}^N and $H \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N)$, $H \geq 0$.

The case where H depends only on u , has been intensively studied, especially the following typical equation

$$(1.2) \quad -\Delta u + |u|^p \text{sign } u = 0$$

with $p > 1$ (see Dynkin [5, 6], Le Gall [9], Gmira and Véron [8], Marcus and Véron [17, 19, 21], Marcus[15] and the references therein). In [8] it was shown that (1.2) admits a *critical value*

$$(1.3) \quad p_c = \frac{N+1}{N-1}.$$

such that, for $1 < p < p_c$, the boundary value problem

$$(1.4) \quad \begin{cases} -\Delta u + |u|^p \text{sign } u = 0 & \text{in } \Omega \\ u = \mu & \text{on } \partial\Omega \end{cases}$$

has a unique solution for every $\mu \in \mathfrak{M}(\partial\Omega)$ (= space of finite Borel measures on $\partial\Omega$). The boundary data is attained as a weak limit of measures. Moreover isolated boundary singularities of solutions of (1.2) can be completely described. For more general results on positive solutions of (1.2) with singular sets on the boundary see [19, 20]. For a treatment of more general equations (where the absorption term H depends on (x, u)) see [2].

The case where H depends only on $|\nabla u|$ has been recently investigated by P.T. Nguyen and L. Véron [22]. For equations of the form

$$(1.5) \quad -\Delta u + g(|\nabla u|) = 0 \quad \text{in } \Omega$$

they obtained a sufficient conditions on g in order that the boundary value problem for (1.5) with measure boundary data have a solution for every measure in $\mathfrak{M}(\partial\Omega)$. If the nonlinearity is of power type, namely $g(|\nabla u|) = |\nabla u|^q$ with $1 \leq q \leq 2$, they showed that the critical value for (1.5) is

$$(1.6) \quad q_c = \frac{N+1}{N}$$

and, for $1 < q < q_c$, they provided a complete description of the positive solutions with isolated singularities on the boundary. The question of uniqueness for (1.5) and some related equations in subcritical case is treated in Appendix A of the present paper by the second author. The proof is based on a technique of [23] adapted to the present case.

Notice that when $q > 2$, by [12] if $u \in C^2(\Omega)$ is a positive solution of (1.5) then u is bounded in Ω . Therefore solutions may exist only for boundary data represented by a bounded function.

In the present paper, we study boundary value problems and boundary singularities of positive solutions of (1.1) when H depends on both u and ∇u . It is convenient to use the following notation: $H \circ u$ is the function given by

$$(H \circ u)(x) = H(x, u(x), \nabla u(x)).$$

We study the case of subquadratic growth in the gradient and concentrate on two model cases:

$$(1.7) \quad H(x, t, \xi) = t^p |\xi|^q$$

where $p > 0$, $0 \leq q \leq 2$ and

$$(1.8) \quad H(x, t, \xi) = t^p + |\xi|^q$$

where $p \geq 1$, $1 \leq q \leq 2$.

Equation (1.1) with H as in (1.8) was studied in [1], [3]; existence and uniqueness of large solutions was established when $1 < p < q \leq 2$. When H is given by (1.7), there exists no large solution of equation (1.1). To our knowledge, up to now, there is no publication treating boundary value problems with measure data for (1.1) and H as in (1.7) or (1.8).

The main difficulty that one encounters in the study of these problems: the inequality $u \leq v$ does not imply any relation between $|\nabla u|$ and $|\nabla v|$. Moreover, in general, the sum of two supersolutions of (1.1) is not a supersolution. In addition, for H as in (1.7) there is no a priori estimate of solutions of (1.1) or of their gradient. (However an upper estimate is available for families of solutions satisfying certain auxiliary conditions.) On the other hand, when H satisfies (1.7) equation (1.1) admits a similarity transformation; when H is as in (1.8), the equation does not admit a similarity transformation unless $p = \frac{q}{2-q}$.

Before stating our main results we introduce some definitions.

Definition 1.1. (i) A function u is a (weak) solution of (1.1) if $u \in L^1_{loc}(\Omega)$, $H \circ u \in L^1_{loc}(\Omega)$ and u satisfies (1.1) in the sense of distribution.

(ii) Let $\mu \in \mathfrak{M}(\partial\Omega)$. A function u is a solution of

$$(1.9) \quad \begin{cases} -\Delta u + H \circ u = 0 & \text{in } \Omega \\ u = \mu & \text{on } \partial\Omega \end{cases}$$

if u satisfies the equation and has boundary trace μ (see Definition 3.6).

Remark. It can be shown that (see Theorem 3.7 below) u is a solution of (1.9) if and only if $u \in L^1(\Omega)$, $H \circ u \in L^1_\rho(\Omega)$ and u satisfies

$$(1.10) \quad \int_{\Omega} (-u \Delta \zeta + (H \circ u) \zeta) dx = - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\mu \quad \forall \zeta \in C^2_0(\overline{\Omega})$$

where $\rho(x) = \text{dist}(x, \partial\Omega)$, \mathbf{n} denotes the outward normal unit vector on $\partial\Omega$ and $C_0^2(\overline{\Omega}) = \{u \in C^2(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$.

Definition 1.2. A positive solution of (1.1) is moderate if $H \circ u \in L_\rho^1(\Omega)$.

Definition 1.3. A nonlinearity H is called subcritical if the problem (1.9) admits a solution for every positive bounded measure μ on $\partial\Omega$. Otherwise, H is called supercritical.

Put

$$(1.11) \quad m_{p,q} := \max \left\{ p, \frac{q}{2-q} \right\}.$$

The first theorem provides a sufficient condition for H to be subcritical and a stability result relative to weak convergence of data. As shown later on (see Theorem F) the sufficient condition is also necessary for subcriticality of H .

Theorem A. Assume either H satisfies (1.7) with $0 < N(p+q-1) < p+1$ or (1.8) with $m_{p,q} < p_c$. Then H is subcritical and the following stability result holds:

Let $\{\mu_n\}$ be a sequence of positive finite measures on $\partial\Omega$ converging weakly to a positive finite measure μ and $\{u_{\mu_n}\}$ be a sequence of corresponding solutions of (1.9) with $\mu = \mu_n$. Then there exists a subsequence such that $\{u_{\mu_{n_k}}\}$ converges to a solution u_μ of (1.9) in $L^1(\Omega)$ and $\{H \circ u_{\mu_{n_k}}\}$ converges to $H \circ u$ in $L_\rho^1(\Omega)$.

Remark. The method of proof of this theorem is classical. It is based on estimates in weak L^p space and compactness of approximating solutions. The results stated in Theorem A can be extended, in the same way, to the following cases:

$$(1.12) \quad 0 \leq H(x, t, \xi) \leq a_1(x)t^p|\xi|^q \quad \forall (x, t, \xi) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^N$$

where $p \geq 0$, $q \geq 0$, $0 < N(p+q-1) < p+1$, $a_1 \in L^\infty(\Omega)$ and $a_1 > c > 0$;

$$(1.13) \quad 0 \leq H(x, t, \xi) \leq a_2(x)f(t) + a_3(x)g(|\xi|) \quad \forall (x, t, \xi) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^N$$

where $a_i \in L^\infty(\Omega)$, $a_i > c > 0$ ($i = 2, 3$), f and g are positive, nondecreasing, continuous functions in \mathbb{R}_+ , satisfying $f(0) = g(0) = 0$ and

$$\int_1^\infty t^{-\frac{2N}{N-1}} f(t) dt < \infty, \quad \int_1^\infty t^{-\frac{2N+1}{N}} g(t) dt < \infty.$$

The next theorem presents an uniqueness result when H satisfies (1.8).

Theorem B. Assume that H satisfies (1.8) and $m_{p,q} < p_c$. Then (1.9) has a unique solution for every $\mu \in \mathfrak{M}^+(\partial\Omega)$.

The uniqueness of solutions of problem (1.9) when H satisfies (1.7) ($0 < N(p+q-1) < p+1$) remains open. However we establish uniqueness in the case that μ is concentrated at a point.

In the next theorems we discuss solutions with an isolated singularity at a point $A \in \partial\Omega$. Without loss of generality we assume that A is the origin.

Theorem C. *Let H be as in Theorem A. Then for any $k > 0$, there exists a unique positive solution of (1.9) with $\mu = k\delta_0$ (where δ_0 is the Dirac mass at the origin). This solution is denoted by $u_{k,0}^\Omega$.*

Furthermore,

$$(1.14) \quad u_{k,0}^\Omega(x) = kP^\Omega(x, 0)(1 + o(1)) \quad \text{as } x \rightarrow 0.$$

and there exists $d_k > 0$ such that

$$(1.15) \quad d_k P^\Omega(x, 0) < u_{k,0}^\Omega(x) < kP^\Omega(x, 0) \quad \forall x \in \Omega.$$

Obviously, $u_{k,0}^\Omega$ is a moderate solution and is called a *weakly singular solution*. It follows from (1.14) that the sequence $\{u_{k,0}^\Omega\}$ is increasing. Moreover, this sequence is uniformly bounded in any compact subset of Ω . Therefore

$$u_{\infty,0}^\Omega := \lim u_{k,0}^\Omega$$

is a solution of (1.1). Clearly this solution is not moderate.

When there is no danger of confusion we drop the upper index writing simply $u_{k,0}$ and $u_{\infty,0}$.

Denote by \mathcal{U}_0^Ω the family of positive *non-moderate* solutions of (1.1) such that $u \in C(\overline{\Omega} \setminus \{0\})$ and $u = 0$ on $\partial\Omega \setminus \{0\}$. If u is such a solution we say that it is a *strongly singular solution*. In the next two theorems we consider solutions of this type.

Theorem D. *Under the assumptions of theorem C, $u_{\infty,0}^\Omega \in \mathcal{U}_0^\Omega$. Furthermore $u_{\infty,0}^\Omega$ is the minimal element of \mathcal{U}_0^Ω .*

Let S^{N-1} be the unit sphere, $\mathbb{R}_+^N = [x_N > 0]$, $S_+^{N-1} = S^{N-1} \cap \mathbb{R}_+^N$ the upper hemisphere and $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$ the spherical coordinates in \mathbb{R}^N . Denote by ∇' and Δ' the covariant derivative on S^{N-1} identified with the tangential derivative and the Laplace-Beltrami operator on S^{N-1} respectively.

As mentioned before, if H is as in (1.7), (1.1) admits a similarity transformation. However, there is no similarity transformation when H satisfies (1.8) unless $p = \frac{q}{2-q}$. When $p \neq \frac{q}{2-q}$ there is competition between u^p and $|\nabla u|^q$. When $p > \frac{q}{2-q}$ the dominant term is u^p ; when $p < \frac{q}{2-q}$ the dominant term is $|\nabla u|^q$. This fact is reflected in the next theorem.

We assume that the set of coordinates is placed so that $0 \in \partial\Omega$, $x_N = 0$ is tangent to $\partial\Omega$ at 0 and the positive x_N axis points into the domain.

Theorem E. *Assume that either H satisfies (1.7), $0 < N(p + q - 1) < p + 1$ and $p \geq 1$ or H satisfies (1.8) and $m_{p,q} < p_c$ where p_c and $m_{p,q}$ are given by (1.3) and (1.11) respectively. Then:*

(i) \mathcal{U}_0^Ω consists of a single element $u_{\infty,0}^\Omega$. In other words, $u_{\infty,0}^\Omega$ is the unique strongly singular solution of (1.1) with singularity at 0.

(ii) Put $r = |x|$, $\sigma = \frac{x}{r}$. Then

$$(1.16) \quad \lim_{x \in \Omega, r \rightarrow 0} r^\beta u_{\infty,0}^\Omega(x) = \omega(\sigma)$$

locally uniformly on S_+^{N-1} where

$$(1.17) \quad \beta = \beta_1 := \frac{2-q}{p+q-1} \quad \text{if } H \text{ satisfies (1.7),}$$

and

$$(1.18) \quad \beta = \beta_2 := \frac{2}{m_{p,q}-1} \quad \text{if } H \text{ satisfies (1.8).}$$

The function ω is the unique solution of the problem

$$(1.19) \quad \begin{cases} -\Delta' \omega + F(\omega, \nabla' \omega) = 0 & \text{in } S^{N-1} \\ \omega = 0 & \text{on } \partial S^{N-1} \end{cases}$$

where $F = F_i(s, \xi)$ ($i = 1, \dots, 4$), $(s, \xi) \in \mathbb{R}_+ \times S^{N-1}$, is given by,

$$(1.20) \quad \begin{aligned} F_1(s, \xi) &= s^p (\beta_1^2 s^2 + |\xi|^2)^{\frac{q}{2}} - \beta_1 (\beta_1 + 2 - N)s, & \text{when } H \text{ satisfies (1.7)} \\ &\text{and, when } H \text{ satisfies (1.8),} \\ F_2(s, \xi) &= s^p + (\beta_2^2 s^2 + |\xi|^2)^{\frac{q}{2}} - \beta_2 (\beta_2 + 2 - N)s, & \text{if } p = \frac{q}{2-q} \\ F_3(s, \xi) &= s^p - \beta_2 (\beta_2 + 2 - N)s, & \text{if } p > \frac{q}{2-q} \\ F_4(s, \xi) &= (\beta_2^2 s^2 + |\xi|^2)^{\frac{q}{2}} - \beta_2 (\beta_2 + 2 - N)s. & \text{if } p < \frac{q}{2-q} \end{aligned}$$

The unique solution of (1.19) with $F = F_i$ will be denoted by ω_i , $i = 1, \dots, 4$. When $i = 1, 2$ we actually have $u_{\infty,0}^{\mathbb{R}_+^N}(x) = r^{-\beta_i} \omega_i(\sigma)$.

Remark. We note that $r^{-\beta_2} \omega_3$ (resp. $r^{-\beta_2} \omega_4$) behaves near the origin like the corresponding strongly singular solution of $-\Delta u + u^p = 0$ (resp. $-\Delta u + |\nabla u|^q = 0$).

Next we present a removability result which implies that the conditions on p, q for H to be subcritical are sharp.

Theorem F Assume that H satisfies either (1.7) with $N(p+q-1) \geq p+1$ or (1.8) with $m_{p,q} \geq p_c$. If $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ is a nonnegative solution of (1.1) vanishing on $\partial\Omega \setminus \{0\}$ then $u \equiv 0$.

When H satisfies (1.8), the removability result is based on the corresponding results for (1.1) with $H = u^p$ and $H = |\nabla u|^q$.

When H satisfies (1.7) and $N(p+q-1) > p+1$ we use a similarity transformation to show that there is no solution with isolated singularity. The case $N(p+q-1) = p+1$

is a bit more delicate. We first establish the removability result for the half-space and use it, together with some regularity results up to the boundary (see [10]) to derive the result in bounded domains of class C^2 .

If $q = 2$, one can obtain removability result by a change of unknown.

Remark. Theorems C, F and G provide a *complete characterization of the positive solutions u of (1.1) in Ω such that $u = 0$ on $\partial\Omega$ except at one point.*

In the case where H satisfies (1.8) we also consider solutions that blow-up strongly on an arbitrary compact set $K \subset \partial\Omega$.

Theorem G *Assume H satisfies (1.8) and $m_{p,q} \leq p_c$ where $m_{p,q}$ and p_c are given by (1.11) and (1.3) respectively. Let K be a compact subset of $\partial\Omega$. Denote by \mathcal{U}_K the family of all positive solutions u of (1.1) such that $\mathcal{S}(u) = K$ (see Definition 3.8) and $u = 0$ on $\Omega \setminus K$. Then there exist a minimal element u_K and a maximal element U_K of \mathcal{U}_K in the sense that $u_K \leq u \leq U_K$ for every $u \in \mathcal{U}_K$. Moreover, for every $y \in K$ and $\gamma \in (0, 1)$, there exist r depending on γ and C depending on N, p, q, γ and the C^2 characteristic of Ω such that*

$$(1.21) \quad U_K(x) \leq C u_K(x) \quad \forall x \in C_{\gamma,r}(y) := \{x \in \Omega : \rho(x) \geq \gamma|x - y|\} \cap B_r(y).$$

This extends a result of [3] on existence of large solutions.

The paper is organized as follows. In section 2, we establish some estimates on positive solution of (1.1) and its gradient, and recall some estimates concerning weak L^p space. Section 3 is devoted to the proof of Theorems A, B and various results on *boundary trace*. In section 4, we provide a complete description of isolated singularities on the boundary (Theorems C,D,E). Boundary value problem with unbounded measure data for (1.1) and H as in (1.8) is discussed in Section 5 (Theorem G). In section 6, we demonstrate the removability result in the supercritical case (Theorem F). In the appendix, a uniqueness result for a class of quasilinear elliptic equations is proved.

2. PRELIMINARIES

Throughout the present paper, we denote by c, c_1, c_2, C, \dots positive constants which may vary from line to line. If necessary the dependence of these constants will be made precise. The following comparison principle can be found in [7, Theorem 9.2].

Proposition 2.1. *Assume $H : D \times \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \mathbb{R}_+$ is nondecreasing with respect to u for any $(x, \xi) \in D \times \mathbb{R}^N$, continuously differentiable with respect to ξ and $H(x, 0, 0) = 0$. Let $u_1, u_2 \in C^2(D) \cap C(\overline{D})$ be two nonnegative solutions of (1.1). If*

$$-\Delta u_1 + H \circ u_1 \leq -\Delta u_2 + H \circ u_2 \quad \text{in } D$$

and $u_1 \leq u_2$ on ∂D . Then $u_1 \leq u_2$ in D .

Next, for $\delta > 0$, we set

$$\Omega_\delta = \{x \in \Omega : \rho(x) < \delta\}, \quad D_\delta = \{x \in \Omega : \rho(x) > \delta\}, \quad \Sigma_\delta = \{x \in \Omega : \rho(x) = \delta\}.$$

Proposition 2.2. *There exists $\delta_0 > 0$ such that*

(i) *For every point $x \in \overline{\Omega}_{\delta_0}$, there exists a unique point $\sigma_x \in \partial\Omega$ such that $|x - \sigma_x| = \rho(x)$. This implies $x = \sigma_x - \rho(x)\mathbf{n}_{\sigma_x}$.*

(ii) *The mappings $x \mapsto \rho(x)$ and $x \mapsto \sigma_x$ belong to $C^2(\overline{\Omega}_{\delta_0})$ and $C^1(\overline{\Omega}_{\delta_0})$ respectively. Furthermore, $\lim_{x \rightarrow \sigma_x} \nabla \rho(x) = -\mathbf{n}_{\sigma_x}$.*

In the sequel, we can assume that $\delta_0 < \|\Delta\rho\|_{L^\infty(\Omega)}^{-1}$. The next results provide a-priori estimates on positive solutions and their gradient.

Proposition 2.3. *Assume H satisfies (1.7) with $p \geq 0$, $0 \leq q < 2$, $p + q > 1$. Let $u \in C^2(\Omega)$ be a positive solution of equation (1.1). Then*

$$(2.1) \quad u(x) \leq \Lambda_1 \rho(x)^{-\beta_1} + \Lambda'_1 \|u\|_{L^1(D_{\frac{\delta_0}{2}})} \quad \forall x \in \Omega,$$

$$(2.2) \quad |\nabla u(x)| \leq \tilde{\Lambda}_1 \rho(x)^{-\beta_1-1} \quad \forall x \in \Omega$$

where β_1 is defined in (1.17), $\tilde{\Lambda}_1 = \tilde{\Lambda}_1(N, p, q, \delta_0, \|u\|_{L^1(D_{\frac{\delta_0}{2}})})$, $\Lambda'_1 = \Lambda'_1(N, \delta_0)$, and

$$(2.3) \quad \Lambda_1 = \left(\frac{\beta_1 + 2}{\beta_1^{q-1}} \right)^{\frac{1}{p+q-1}}.$$

Proof. Put $M_{\delta_0} = \max\{u(x) : x \in \overline{D}_{\delta_0}\}$. For each $\delta \in (0, \delta_0)$, we set

$$w_\delta(x) = \Lambda_1(\rho(x) - \delta)^{-\beta_1} + M_{\delta_0} \quad x \in D_\delta.$$

By a simple computation, we obtain

$$-\Delta w_\delta + w_\delta^p |\nabla w_\delta|^q > 0 \quad \text{in } \Omega_{\delta_0} \setminus \overline{\Omega}_\delta.$$

Since $w_\delta \geq u$ on $\Sigma_\delta \cup \Sigma_{\delta_0}$, by the comparison principle Proposition 2.1, $u \leq w_\delta$ in $\Omega_{\delta_0} \setminus \overline{\Omega}_\delta$. Letting $\delta \rightarrow 0$ yields

$$(2.4) \quad u(x) \leq \Lambda_1 \rho(x)^{-\beta_1} + M_{\delta_0} \quad \forall x \in \Omega.$$

Since u is subharmonic, by [25, Theorem 1], there exists $\Lambda'_1 = \Lambda'_1(N, \delta_0)$ such that $\Lambda'_1 \|u\|_{L^1(D_{\delta_0/2})} > M_{\delta_0}$. This, along with (2.4), implies (2.1).

Next we prove (2.2). Fix $x_0 \in \Omega$ and set $d_0 = \frac{1}{3}\rho(x_0)$, $y_0 = \frac{1}{d_0}x_0$ and

$$M_0 = \max\{u(x) : x \in B_{2d_0}(x_0)\}, \quad u_0(y) = \frac{u(x)}{M_0}, \quad y = \frac{1}{d_0}x \in B_2(y_0).$$

Then $\max\{u_0(y) : y \in B_2(y_0)\} = 1$ and $-\Delta u_0 + M_0^{p+q-1} d_0^{2-q} u_0^p |\nabla u_0|^q = 0$ in $B_2(y_0)$. By [14], there exists a positive constant $c = c(N, p, q, \delta_0, \|u\|_{L^1(D_{\frac{\delta_0}{2}})})$ such that $\max_{B_1(y_0)} |\nabla u_0| \leq c$. Consequently,

$$\max_{B_{d_0}(x_0)} |\nabla u| \leq c d_0^{-1} \max_{B_{2d_0}(x_0)} u$$

which implies (2.2). \square

Proposition 2.4. *Assume H satisfies (1.8) with $p > 1$, $1 < q < 2$. Let $u \in C^2(\Omega)$ be a positive solution of equation (1.1). Then*

$$(2.5) \quad u(x) \leq \Lambda_2 \rho(x)^{-\beta_2}$$

$$(2.6) \quad |\nabla u(x)| \leq \tilde{\Lambda}_2 \rho(x)^{-\beta_2-1} \quad \forall x \in \Omega$$

where β_2 is defined in (1.18), $\Lambda_2 = \Lambda_2(N, p, q, \delta_0)$ and $\tilde{\Lambda}_2 = \tilde{\Lambda}_2(N, p, q, \delta_0)$.

Proof. Since u is a subsolution of (1.2), it follows from Keller-Osserman [21] that

$$(2.7) \quad u(x) \leq c \rho(x)^{-\frac{2}{p-1}} \quad \forall x \in \Omega$$

where $c = c(N, p)$. By a similar argument as in the proof of Proposition 2.3, we deduce that

$$(2.8) \quad u(x) \leq c' \rho(x)^{-\beta_2} + c'' \|u\|_{L^1(D_{\frac{\delta_0}{2}})} \quad \forall x \in \Omega$$

where $c' = c'(p, q)$ and $c'' = c''(N, \delta_0)$. Combining (2.7) and (2.8) implies (2.5). Finally, we derive (2.6) from Proposition 2.3 as in the proof of Proposition 2.3. \square

Denote by G^Ω (resp. P^Ω) the Green kernel (resp. the Poisson kernel) in Ω , with corresponding operators \mathbb{G}^Ω (resp. \mathbb{P}^Ω). We denote by $\mathfrak{M}_{\rho^\alpha}(\Omega)$, $\alpha \in [0, 1]$, the space of Radon measures μ on Ω satisfying $\int_\Omega \rho^\alpha d|\mu| < \infty$, by $\mathfrak{M}(\partial\Omega)$ the space of bounded Radon measures on $\partial\Omega$ and by $\mathfrak{M}^+(\partial\Omega)$ the positive cone of $\mathfrak{M}(\partial\Omega)$.

Denote $L_w^p(\Omega; \tau)$, $1 \leq p < \infty$, $\tau \in \mathfrak{M}^+(\Omega)$, the weak L^p space defined as follows: a measurable function f in Ω belongs to this space if there exists a constant c such that

$$(2.9) \quad \lambda_f(a; \tau) := \tau(\{x \in \Omega : |f(x)| > a\}) \leq c a^{-p}, \quad \forall a > 0.$$

The function λ_f is called the distribution function of f (relative to τ). For $p \geq 1$, denote

$$L_w^p(\Omega; \tau) = \{f \text{ Borel measurable} : \sup_{a>0} a^p \lambda_f(a; \tau) < \infty\}$$

and

$$(2.10) \quad \|f\|_{L_w^p(\Omega; \tau)}^* = \left(\sup_{a>0} a^p \lambda_f(a; \tau) \right)^{\frac{1}{p}}.$$

The $\|\cdot\|_{L_w^p(\Omega;\tau)}$ is not a norm, but for $p > 1$, it is equivalent to the norm

$$(2.11) \quad \|f\|_{L_w^p(\Omega;\tau)} = \sup \left\{ \frac{\int_\omega |f| d\tau}{\tau(\omega)^{1/p'}} : \omega \subset \Omega, \omega \text{ measurable}, 0 < \tau(\omega) < \infty \right\}.$$

More precisely,

$$(2.12) \quad \|f\|_{L_w^p(\Omega;\tau)}^* \leq \|f\|_{L_w^p(\Omega;\tau)} \leq \frac{p}{p-1} \|f\|_{L_w^p(\Omega;\tau)}^*$$

Notice that, for every $\alpha > -1$,

$$L_w^p(\Omega; \rho^\alpha dx) \subset L_{\rho^\alpha}^s(\Omega) \quad \forall s \in [1, p).$$

The following useful estimates involving Green and Poisson operators can be found in [4] (see also [21] and [26]).

Proposition 2.5. *For any $\alpha \in [0, 1]$, there exist a positive constant c_1 depending on α , Ω and N such that*

$$(2.13) \quad \|\mathbb{G}^\Omega[\nu]\|_{L_w^{\frac{N+\alpha}{N+\alpha-2}}(\Omega; \rho^\alpha dx)} + \|\nabla \mathbb{G}^\Omega[\nu]\|_{L_w^{\frac{N+\alpha}{N+\alpha-1}}(\Omega; \rho^\alpha dx)} \leq c_1 \|\nu\|_{\mathfrak{M}_{\rho^\alpha}(\Omega)},$$

$$(2.14) \quad \|\mathbb{P}^\Omega[\mu]\|_{L_w^{\frac{N+\alpha}{N-1}}(\Omega; \rho^\alpha dx)} + \|\nabla \mathbb{P}^\Omega[\mu]\|_{L_w^{\frac{N+1}{N}}(\Omega; \rho^\alpha dx)} \leq c_1 \|\mu\|_{\mathfrak{M}(\partial\Omega)},$$

for any $\nu \in \mathfrak{M}_{\rho^\alpha}(\Omega)$ and any $\mu \in \mathfrak{M}(\partial\Omega)$ where

$$\|\nu\|_{\mathfrak{M}_{\rho^\alpha}(\Omega)} := \int_\Omega \rho^\alpha d|\nu| \quad \text{and} \quad \|\mu\|_{\mathfrak{M}(\partial\Omega)} = \int_{\partial\Omega} d|\mu|.$$

3. BOUNDARY VALUE PROBLEM WITH MEASURES AND BOUNDARY TRACE

3.1. The Dirichlet problem. Proof of Theorem A. We deal with the case when H satisfies (1.7). The case H satisfies (1.8) is simpler and can be treated in a similar way.

Let $\{\mu_n\}$ be a sequence of positive functions in $C^1(\partial\Omega)$ converging weakly to μ . There exists a positive constant c_2 independent of n such that $\|\mu_n\|_{L^1(\partial\Omega)} \leq c_2 \|\mu\|_{\mathfrak{M}(\partial\Omega)}$ for all n . Consider the following problem

$$(3.1) \quad \begin{cases} -\Delta v + (v + \mathbb{P}^\Omega[\mu_n])^p |\nabla(v + \mathbb{P}^\Omega[\mu_n])|^q = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Since 0 and $-\mathbb{P}^\Omega[\mu_n]$ are respectively supersolution and subsolution of (3.1), by [13, Theorem 6.5] there exists a solution $v_n \in W^{2,s}(\Omega)$ with $1 < s < \infty$ to problem (3.1) satisfying $-\mathbb{P}^\Omega[\mu_n] \leq v_n \leq 0$. Thus $u_n = v_n + \mathbb{P}^\Omega[\mu_n]$ is a solution of

$$(3.2) \quad \begin{cases} -\Delta u_n + u_n^p |\nabla u_n|^q = 0 & \text{in } \Omega \\ u_n = \mu_n & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle, such solution is the unique solution of (3.2).

Assertion 1: $\{u_n\}$ and $\{|\nabla u_n|\}$ remain uniformly bounded respectively in $L_w^{\frac{N}{N-1}}(\Omega)$ and $L_w^{\frac{N+1}{N}}(\Omega; \rho dx)$.

Let ξ be the solution to

$$(3.3) \quad -\Delta \xi = 1 \text{ in } \Omega, \quad \xi = 0 \text{ on } \partial\Omega,$$

then there exists a constant $c_3 > 0$ such that $c_3^{-1} < -\frac{\partial \xi}{\partial \mathbf{n}} < c_3$ on $\partial\Omega$ and $c_3^{-1}\rho \leq \xi \leq c_3\rho$ in Ω . By multiplying the equation in (3.2) by ξ and integrating on Ω , we obtain

$$(3.4) \quad \int_{\Omega} u_n dx + \int_{\Omega} u_n^p |\nabla u_n|^q \rho dx \leq c_4 \|\mu\|_{\mathfrak{M}(\partial\Omega)}$$

where c_4 is a positive constant independent of n . From Proposition 2.5 and by noticing that $u_n \leq \mathbb{P}^{\Omega}[\mu_n]$, for every $\alpha \in [0, 1]$, we get

$$(3.5) \quad \|u_n\|_{L_w^{\frac{N+\alpha}{N-1}}(\Omega; \rho^{\alpha} dx)} \leq \|\mathbb{P}^{\Omega}[\mu_n]\|_{L_w^{\frac{N+\alpha}{N-1}}(\Omega; \rho^{\alpha} dx)} \leq c_1 \|\mu_n\|_{L^1(\partial\Omega)} \leq c_1 c_2 \|\mu\|_{\mathfrak{M}(\partial\Omega)}.$$

Again, from Proposition 2.5 and (3.4), we derive that

$$(3.6) \quad \|\nabla u_n\|_{L_w^{\frac{N+1}{N}}(\Omega; \rho dx)} \leq c_1 \left(\|u_n^p |\nabla u_n|^q\|_{L_{\rho}^1(\Omega)} + \|\mu_n\|_{L^1(\partial\Omega)} \right) \leq c_5 \|\mu\|_{\mathfrak{M}(\partial\Omega)}$$

where c_5 is a positive constant depending only on Ω and N . Thus Assertion 1 follows from (3.5) and (3.6).

By regularity results for elliptic equations [16], there exist a subsequence, still denoted by $\{u_n\}$, and a function u such that $\{u_n\}$ and $\{|\nabla u_n|\}$ converges to u and $|\nabla u|$ a.e. in Ω .

Assertion 2: $\{u_n\}$ converges to u in $L^1(\Omega)$.

Indeed, by taking $\alpha = 0$ in (3.5), we derive $\{u_n\}$ is uniformly bounded in $L_w^{\frac{N}{N-1}}(\Omega)$. Therefore, $\{u_n\}$ is uniformly bounded in $L^r(\Omega)$ for any $r \in [1, \frac{N}{N-1})$. By Holder inequality, $\{u_n\}$ is uniformly integrable in $L^1(\Omega)$. Thus Assertion 2 follows from Vitali's convergence theorem.

Assertion 3: $\{u_n^p |\nabla u_n|^q\}$ converges to $u^p |\nabla u|^q$ in $L_{\rho}^1(\Omega)$.

Indeed, by taking $\alpha = 1$ in (3.5), one derives that $\{u_n\}$ is uniformly bounded in $L_w^{\frac{N+1}{N}}(\Omega; \rho dx)$. Therefore, $\{u_n\}$ is uniformly bounded in $L_{\rho}^r(\Omega)$ for every $r \in [1, \frac{N+1}{N-1})$. By (3.6), $\{|\nabla u_n|\}$ is uniformly bounded in $L_{\rho}^s(\Omega)$ for every $s \in [1, \frac{N+1}{N})$. Since $N(p+q-1) < p+1$, we can choose r and s close to $\frac{N+1}{N-1}$ and $\frac{N+1}{N}$ respectively so that $\frac{p}{r} + \frac{q}{s} < 1$. By Holder inequality, $\{u_n^p |\nabla u_n|^q\}$ is uniformly integrable in $L_{\rho}^1(\Omega)$. Thus Assertion 3 follows from Vitali's convergence theorem.

For every $\zeta \in C_0^2(\overline{\Omega})$, we have

$$(3.7) \quad \int_{\Omega} (-u_n \Delta \zeta + u_n^p |\nabla u_n|^q \zeta) dx = - \int_{\partial\Omega} \mu_n \frac{\partial \zeta}{\partial \mathbf{n}} dS.$$

Due to Assertions 2 and 3, by letting $n \rightarrow \infty$ in (3.7) we obtain (1.10); so u is a solution of (1.9). By Proposition 2.5, $u \in L_w^{\frac{N}{N-1}}(\Omega)$ and $|\nabla u| \in L_w^{\frac{N+1}{N}}(\Omega; \rho dx)$.

Next, let $\{\mu_n\}$ be a sequence of positive finite measures on $\partial\Omega$ which converges weakly to a positive finite measure μ and $\{u_{\mu_n}\}$ be a sequence of corresponding solutions of (3.2). Then by using a similar argument as in Assertions 2 and 3, we deduce that there exists a subsequence such that $\{u_{\mu_{n_k}}\}$ converges to a solution u_μ of (1.9) in $L^1(\Omega)$ and $\{H \circ u_{\mu_{n_k}}\}$ converges to $H \circ u$ in $L_\rho^1(\Omega)$. \square

Using Theorem A one can establish a slightly stronger type of stability.

Corollary 3.1. *Let H be as in theorem A. Let $\{a_n\}$ be a decreasing sequence converging to 0, μ be a bounded positive measure on $\partial\Omega$ and $\{\mu_n\}$ be a sequence of bounded positive measure on Σ_{a_n} converging weakly to μ . Let $\{u_{\mu_n}\}$ be a sequence of corresponding solutions of (3.2) in D_{a_n} . Then there exists a subsequence such that $\{u_{\mu_{n_k}}\}$ converges in $L^1(\Omega)$ to a solution u_μ of (1.9) and $\{H \circ u_{\mu_{n_k}}\}$ converges to $H \circ u$ in $L_\rho^1(\Omega)$.*

Proof. As above, we consider the case H satisfies (1.7) because the case H satisfies (1.8) can be proved by a similar argument. We extend u_{μ_n} and $|\nabla u_{\mu_n}|$ by zero outside \overline{D}_{a_n} and still denote them by the same expressions. By regularity results for elliptic equations [16], there exist a subsequence, still denoted by $\{u_{\mu_n}\}$, and a function u such that $\{u_{\mu_n}\}$ and $\{|\nabla u_{\mu_n}|\}$ converges to u and $|\nabla u|$ a.e. in Ω . Let $G \subset \Omega$ be a Borel set and put $G_n = G \cap D_{a_n}$. By using similar argument as in Assertion 2 in the proof of theorem A, due to the estimate $\|\mathbb{P}^\Omega[\mu]\|_{L^1(\Sigma_{a_n})} \leq c_7 \|\mu\|_{\mathfrak{M}(\Sigma)}$, we derive

$$(3.8) \quad \begin{aligned} \int_{G_n} u_{\mu_n} dx &\leq |G_n|^{\frac{1}{N}} \|u_{\mu_n}\|_{L_w^{\frac{N}{N-1}}(D_{a_n})} \leq c_1 c_2 |G_n|^{\frac{1}{N}} \left\| \mathbb{P}^\Omega[\mu] \right\|_{L^1(\Sigma_{a_n})} \\ &\leq c_1 c_2 c_7 |G|^{\frac{1}{N}} \|\mu\|_{\mathfrak{M}(\Sigma)}. \end{aligned}$$

Hence $\{u_{\mu_n}\}$ is uniformly integrable. Therefore due to Vitali's convergence theorem, up to a subsequence, $\{u_{\mu_n}\}$ converges to u in $L^1(\Omega)$.

Set $\rho_n(x) := (\rho(x) - a_n)_+$. By proceeding as in Assertion 3 of the proof of Theorem A and notice that $\int_{G_n} \rho_n dx \leq \int_G \rho dx$, we derive that $\{u_{\mu_n}^p |\nabla u_{\mu_n}|^q\}$ is uniformly integrable. Therefore by Vitali's convergence, up to a subsequence, $\{u_{\mu_n}^p |\nabla u_{\mu_n}|^q\}$ converges to $u^p |\nabla u|^q$ in $L_\rho^1(\Omega)$.

Finally, if $\zeta \in C_0^2(\overline{\Omega})$ we denote by ζ_n the solution of

$$(3.9) \quad -\Delta \zeta_n = -\Delta \zeta \text{ in } D_{a_n}, \quad \zeta_n = 0 \text{ on } \partial D_{a_n}.$$

Then $\zeta_n \in C_0^2(\overline{\Omega}_{a_n})$, $\zeta_n \rightarrow \zeta$ in $C^2(\Omega)$ and $\sup_n \|\zeta_n\|_{C^2(\overline{\Omega}_{a_n})} < \infty$. Since

$$(3.10) \quad \int_{D_{a_n}} (-u_{\mu_n} \Delta \zeta_n + u_{\mu_n}^p |\nabla u_{\mu_n}|^q \zeta_n) dx = - \int_{\Sigma_{a_n}} \frac{\partial \zeta_n}{\partial \mathbf{n}} d\mu_n,$$

by letting $n \rightarrow \infty$, we deduce that u is a solution of (1.9). \square

Remark. Let $\mu \in \mathfrak{M}^+(\partial\Omega)$. It follows from Proposition 2.3 and Proposition 2.4 that there exists a constant c depending on N, p, q, Ω and $\|\mu\|_{\mathfrak{M}(\partial\Omega)}$ such that for every positive solution u of (1.9) there holds

$$(3.11) \quad u(x) \leq c\rho(x)^{-\beta_i} \quad \forall x \in \Omega,$$

$$(3.12) \quad |\nabla u(x)| \leq c\rho(x)^{-\beta_i-1} \quad \forall x \in \Omega$$

where

$$i = \begin{cases} 1 & \text{if } H \text{ satisfies (1.7)} \\ 2 & \text{if } H \text{ satisfies (1.8)}. \end{cases}$$

Using these facts and Corollary 3.1 we obtain the following monotonicity result.

Corollary 3.2. *Let H be as in theorem A. For any $\mu \in \mathfrak{M}^+(\partial\Omega)$, there exists a maximal solution U_μ of (1.9). Moreover, if $\mu, \nu \in \mathfrak{M}^+(\partial\Omega)$ such that $\mu \leq \nu$ then $U_\mu \leq U_\nu$.*

Proof. For each $\delta > 0$, let $U := U_{\mu,\delta}$ be the solution of

$$(3.13) \quad \begin{cases} -\Delta U + H \circ U = 0 & \text{in } D_\delta \\ U = \mathbb{P}^\Omega[\mu] & \text{on } \Sigma_\delta. \end{cases}$$

By the comparison principle, $0 \leq U_{\mu,\delta} \leq \mathbb{P}^\Omega[\mu]$, hence $\{U_{\mu,\delta}\}$ is decreasing as $\delta \rightarrow 0$. Put $U_\mu := \lim_{\delta \rightarrow 0} U_{\mu,\delta}$ then by Corollary 3.1 U_μ is a solution of (1.9). If u is a positive solution of (1.9) then by the comparison principle $0 \leq u \leq \mathbb{P}^\Omega[\mu]$ in Ω . Therefore $u \leq U_{\mu,\delta}$ in D_δ for every $\delta > 0$. Letting $\delta \rightarrow 0$ implies $u \leq U_\mu$.

Next, if $\mu \leq \nu$ then $\mathbb{P}^\Omega[\mu] \leq \mathbb{P}^\Omega[\nu]$. Hence $U_{\mu,\delta} \leq U_{\nu,\delta}$ for every $\delta > 0$ and therefore $U_\mu \leq U_\nu$. \square

Proof of Theorem B. The strategy is the same as in the proof of Theorem A.1 so we only sketch the main technical modifications. Let u be a positive solution of (1.9) then $u \leq U_\mu$. Let $\{\mu_n\}$ be a sequence of functions in $C^1(\partial\Omega)$ converging weakly to μ . For $k > 0$, denote by T_k the truncation function, i.e. $T_k(s) = \max(-k, \min(s, k))$. For every $n > 0$, denote by u_n and $U_{\mu,n}$ respectively the solutions of

$$(3.14) \quad -\Delta u_n + T_n(H \circ u) = 0 \quad \text{in } \Omega, \quad u_n = \mu_n \quad \text{on } \partial\Omega.$$

$$(3.15) \quad -\Delta U_{\mu,n} + T_n(H \circ U_\mu) = 0 \quad \text{in } \Omega, \quad U_{\mu,n} = \mu_n \quad \text{on } \partial\Omega.$$

By local regularity theory for elliptic equations (see, e.g., [16]), $u_n \rightarrow u$ and $U_{\mu,n} \rightarrow U_\mu$ in $C_{loc}^1(\Omega)$. From (3.14) and (3.15) we obtain

$$(3.16) \quad \begin{cases} -\Delta(U_{\mu,n} - u_n) = -T_n(H \circ U_\mu) + T_n(H \circ u) & \text{in } \Omega \\ U_{\mu,n} - u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

We shall prove that $U_\mu = u$. By contradiction, we assume that $M := \sup_\Omega (U_\mu - u) \in (0, \infty]$. Let $0 < k < M$. From (3.16), Kato's inequality [21] and the fact that $u \leq U_\mu$, we get

$$(3.17) \quad \begin{aligned} -\Delta(U_{\mu,n} - u_n - k)_+ &\leq (T_n(H \circ u) - T_n(H \circ U_{\mu,n}))\chi_{E_{n,k}} \\ &\leq ||\nabla U_\mu|^q - |\nabla u|^q|\chi_{E_{n,k}} \end{aligned}$$

where $E_{n,k} = \{x \in \Omega : u_{1,n} - u_{2,n} > k\}$. We next proceed as in the proof of Theorem A.1 in order to get a contradiction. Thus $u = U_\mu$. \square

As a consequence, we obtain the following comparison principle

Corollary 3.3. *Under the assumption of Theorem B, if u_1 and u_2 be respectively positive sub and supersolution solution of (1.1) such that $\text{tr}(u_1) \leq \text{tr}(u_2)$ then $u_1 \leq u_2$ in Ω .*

Proof. We first observe that if u_1 and u_2 are both solution of (1.1) then by Theorem A and B, $u_1 \leq u_2$.

Next we consider the case u_1 and u_2 are respectively sub and super solution. For $\delta > 0$, let $v_{i,\delta}$, $i = 1, 2$ be the solution of

$$(3.18) \quad \begin{cases} -\Delta v + H \circ v = 0 & \text{in } D_\delta \\ v = u_i & \text{on } \Sigma_\delta \end{cases}$$

By the comparison principle, $u_1 \leq v_{1,\delta}$ and $v_{2,\delta} \leq u_2$ in D_δ . Therefore $\{v_{1,\delta}\}$ and $\{v_{2,\delta}\}$ are respectively increasing and decreasing as $\delta \rightarrow 0$. By Corollary 3.1 and Theorem B, $v_i := \lim_{\delta \rightarrow 0} v_{i,\delta}$ is the solution of (1.1) with boundary trace μ_i . Moreover, $u_1 \leq v_1$ and $v_2 \leq u_2$. Since $\mu_1 \leq \mu_2$, by the above observation, $v_1 \leq v_2$. Thus $u_1 \leq v_1 \leq v_2 \leq u_2$. \square

When H satisfies (1.7), the question of uniqueness remains open, but we can show that any positive solution of (1.9) behaves like U_μ near the boundary. Before stating the result, we need the following definition

Definition 3.4. *A nonnegative superharmonic function is called a potential if its largest harmonic minorant is zero.*

Proposition 3.5. *Let $\mu \in \mathfrak{M}^+(\partial\Omega)$. If u is a positive solution of (1.9) then*

$$(3.19) \quad \lim_{x \rightarrow y} \frac{u(x)}{\mathbb{P}^\Omega[\mu](x)} = 1 \quad \text{non-tangentially, } \mu - \text{a.e.}$$

Moreover, under the assumptions of theorem A, there holds

$$(3.20) \quad \lim_{x \rightarrow y} \frac{u(x)}{U_\mu(x)} = 1 \quad \text{non-tangentially, } \mu - \text{a.e.}$$

Proof. Put $v_\mu = \mathbb{P}^\Omega[\mu] - u$ then $v_\mu > 0$ and $-\Delta v_\mu = H \circ u \geq 0$ in Ω . It means v_μ is a positive superharmonic function in Ω . By Riesz Representation Theorem (see [15]), v_μ can be written under the form $v_\mu = v_h + v_p$ where v_h is a nonnegative harmonic function and v_p is a potential. Since the boundary trace of v_μ is a zero measure, it follows that the boundary trace of v_h and v_p is zero measure. Therefore $v_h = 0$ in Ω and $v_\mu = v_p$. By [15, Theorem 2.11 and Lemma 2.13], we derive (3.19). Then (3.20) follows straightforward from Corollary 3.2 and (3.19). \square

3.2. Moderate solutions and boundary trace. In this section we study the notion of boundary trace of positive solutions of (1.1). We start with some notations.

Definition 3.6. Let $u \in W_{loc}^{1,s}(\Omega)$ for some $s > 1$. We say that u possesses an M-boundary trace on $\partial\Omega$ if there exists $\mu \in \mathfrak{M}(\partial\Omega)$ such that, for every uniform C^2 exhaustion $\{D_n\}$ (see [21, Definition 1.3.1]) and every $\phi \in C(\overline{\Omega})$,

$$(3.21) \quad \lim_{n \rightarrow \infty} \int_{\partial D_n} u|_{\partial D_n} \phi dS = \int_{\partial\Omega} \phi d\mu.$$

The M-boundary trace of u is denoted by $\text{tr}(u)$.

Let A be a relatively open subset of $\partial\Omega$. A measure $\mu \in \mathfrak{M}(A)$ is the M-boundary trace of u on A if (3.21) holds for every $\phi \in C(\overline{\Omega})$ such that $\text{supp } \phi \subset\subset \Omega \cup A$. In case of positive functions, the definition can be extended to include positive Radon measure on A .

Characterization of moderate solutions (see Definition 1.2) is given in the next result.

Theorem 3.7. Let u be a positive solution of (1.1). Then the following statements are equivalent:

- (i) u is bounded from above by an harmonic function in Ω .
- (ii) u is moderate.
- (iii) u possesses an M-boundary trace denoted by μ
- (iv) u is a solution of (1.9).
- (v) $u \in L^1(\Omega)$, $H \circ u \in L_\rho^1(\Omega)$ and the integral formulation (1.10) holds where $\mu = \text{tr}(u)$.

Proof. (i) \implies (ii). Suppose $u \leq U$ where U is a positive harmonic function. By Herglotz's theorem, U admits an M-boundary trace and therefore

$$\lim_{\delta \rightarrow 0} \int_{\Sigma_\delta} U dS < \infty.$$

It follows that $u \in L^1(\Omega)$ and

$$\sup_{0 < \delta < \delta_0} \int_{\Sigma_\delta} u dS < \infty.$$

Consequently there exist a sequence $\{\delta_n\}$ converging to zero and a measure $\mu \in \mathfrak{M}^+(\partial\Omega)$ such that

$$\lim_{n \rightarrow \infty} \int_{\Sigma_{\delta_n}} u \phi dS = \int_{\partial\Omega} \phi d\mu$$

for every nonnegative function $\phi \in C(\overline{\Omega})$. Since u is a solution of (1.1),

$$(3.22) \quad - \int_{D_\delta} u \Delta \zeta dx + \int_{D_\delta} (H \circ u) \zeta dx = - \int_{\Sigma_\delta} u \frac{\partial \zeta}{\partial \mathbf{n}} dS$$

for every $\zeta \in C_0^2(\overline{D_\delta})$ and $\delta \in (0, \delta_0)$. Given $\phi \in C^2(\overline{\Omega})$, then there exists a sequence $\{\varphi_n\}$ and a function φ such that

$$(3.23) \quad \begin{aligned} & \varphi_n \in C_0^2(\overline{D_{\delta_n}}), \quad \frac{\partial \varphi_n}{\partial \mathbf{n}}|_{\Sigma_{\delta_n}} = \phi, \quad \varphi \in C_0^2(\overline{\Omega}), \quad \frac{\partial \varphi}{\partial \mathbf{n}}|_{\partial\Omega} = \phi, \\ & \|\varphi_n\|_{C^2(\overline{D_{\delta_n}})} < c \|\phi\|_{C^2(\overline{\Omega})}, \quad \varphi_n \leq c \rho_n \phi, \\ & \varphi_n / \rho_n \rightarrow \varphi / \rho \text{ in } C_{loc}^2(\Omega). \end{aligned}$$

The constant c is independent of ϕ and n , but depends on the exhaustion. Consider (3.22) with $\delta = \delta_n$ and $\zeta = \varphi_n$. We see that the first and third terms in (3.22) converge when $n \rightarrow \infty$. Therefore the second term converges and we get

$$(3.24) \quad - \int_{\Omega} u \Delta \varphi dx + \int_{\Omega} (H \circ u) \varphi dx = - \int_{\partial\Omega} \frac{\partial \varphi}{\partial \mathbf{n}} d\mu.$$

By choosing $\phi = 1$ in Ω and $\varphi = \rho$ in $\Omega_{\delta_0/2}$, we deduce that $H \circ u \in L_\rho^1(\Omega)$.

(ii) \implies (iii). Put $v = u + \mathbb{G}^\Omega[H \circ u]$ then v is a positive harmonic function. Therefore v possesses an M-boundary trace μ . Since $\text{tr}(\mathbb{G}^\Omega[H \circ u]) = 0$, it follows that $\text{tr}(u) = \mu$.

(iii) \implies (iv). The implication is obvious.

(iv) \implies (v). Let $\{D_n\}$ be a uniform C^2 exhaustion of Ω . For every n , denote by U_n the harmonic function in D_n such that $U_n = u$ on ∂D_n . By the comparison principle, $u \leq U_n$ on D_n . The sequence $\{U_n\}$ converges to a positive harmonic function U which dominates u in Ω . Since u possesses an M-boundary trace μ , it follows that U admits an M-boundary trace μ . Hence $U \in L^1(\Omega)$ and consequently $u \in L^1(\Omega)$. By proceeding as above, we deduce that $H \circ u \in L_\rho^1(\Omega)$. Let $\phi \in C^2(\overline{\Omega})$ and let φ and $\{\varphi_n\}$ as in (3.23) with D_{δ_n} replaced by D_n . We have

$$- \int_{D_n} u \Delta \varphi_n dx + \int_{D_n} (H \circ u) \varphi_n dx = - \int_{\partial D_n} u \phi dS.$$

As $\{\varphi_n / \rho\}$ and $\{\Delta \varphi_n\}$ are bounded sequences converging to φ / ρ and $\Delta \varphi$ respectively and $\text{tr}(u) = \mu$, by letting $n \rightarrow \infty$, we obtain (1.10).

(v) \implies (i). The implication follows from the estimate $u \leq \mathbb{P}^\Omega[\mu]$ in Ω . \square

Motivated by the above result, we introduce the following definition.

Definition 3.8. *Let u be a positive solution of (1.1). A point $y \in \partial\Omega$ is regular relative to u if there is a neighborhood Q of y such that*

$$\int_{Q \cap \Omega} (H \circ u) \rho dx < \infty.$$

Otherwise we say that y is a singular point relative to u .

The set of regular points is denoted by $\mathcal{R}(u)$, while the set of singular points is denoted by $\mathcal{S}(u)$.

Remark. Clearly $\mathcal{R}(u)$ is relatively open.

The next result can be obtained by combining the argument in the proof of [21, Theorem 3.1.8] and Theorem 3.7.

Theorem 3.9. *Let u be a positive solution of (1.1). Then*

(i) u has an M -boundary trace on $\mathcal{R}(u)$ given by a positive Radon measure μ . Hence

$$\lim_{\delta \rightarrow 0} \int_{\Sigma_\delta} u \phi dS = \int_{\partial\Omega} \phi d\mu$$

for every $\phi \in C(\overline{\Omega})$ such that $\text{supp } \phi \subset \mathcal{R}(u)$.

(ii) A point $y \in \partial\Omega$ is singular relative to u if and only if for every $r > 0$,

$$(3.25) \quad \limsup_{\delta \rightarrow 0} \int_{B_r(y) \cap \Sigma_\delta} u dS = \infty.$$

Remark. From the above results, we see that u is a moderate solution if and only if $\mathcal{S}(u) = \emptyset$.

Next we give some results concerning the minimum and the maximum of two positive solutions.

Lemma 3.10. *Let u_1 and u_2 be two positive solutions of (1.1). Then $\max(u_1, u_2)$ and $\min(u_1, u_2)$ are respectively a subsolution and a supersolution of (1.1). Assume in addition that $\text{tr}(u_i) = \mu_i \in \mathfrak{M}^+(\partial\Omega)$, $i = 1, 2$. Then $\text{tr}(\max(u_1, u_2)) = \max(\mu_1, \mu_2)$ and $\text{tr}(\min(u_1, u_2)) = \min(\mu_1, \mu_2)$.*

Proof. Put $v = \max(u_1, u_2) = (u_1 - u_2)_+ + u_2$. Since $u_i \in W^{1,s}(\Omega)$ for some $s > 1$, it follows that $v \in W^{1,s}(\Omega)$ and

$$(3.26) \quad \nabla v = \begin{cases} \nabla u_1 & \text{if } u_1 > u_2 \\ \nabla u_2 & \text{if } u_1 \leq u_2 \end{cases} \quad \text{a.e. in } \Omega.$$

By Kato's inequality (see [21]),

$$(3.27) \quad \begin{aligned} \Delta v &= \Delta(u_1 - u_2)_+ + \Delta u_2 \leq \text{sign}_+(u_1 - u_2) \Delta(u_1 - u_2) + \Delta u_2 \\ &= \text{sign}_+(u_1 - u_2) (H \circ u_1 - H \circ u_2) + H \circ u_2. \end{aligned}$$

Combining (3.26) and (3.27) implies $-\Delta v + H \circ v \leq 0$ in Ω , namely v is a subsolution of (1.1). Similarly, $\min(u, v)$ is a supersolution of (1.1).

It follows from Theorem 3.7 that $u_i \leq \mathbb{P}^\Omega[\mu_i]$, $i = 1, 2$. Hence

$$v \leq \max(\mathbb{P}^\Omega[\mu_1], \mathbb{P}^\Omega[\mu_2]).$$

Consequently, $\text{tr}(v) = \max(\mu_1, \mu_2)$. Since $\min(u_1, u_2) = u_1 + u_2 - \max(u_1, u_2)$, it follows that

$$\text{tr}(\min(u_1, u_2)) = \mu_1 + \mu_2 - \max(\mu_1, \mu_2) = \min(\mu_1, \mu_2).$$

□

As a consequence, we obtain

Corollary 3.11. *Let u_i , $i = 1, 2$ be positive solutions of (1.1) such that $\text{tr}(u_i) = \mu_i \in \mathfrak{M}^+(\partial\Omega)$. Then there exists a minimal solution \bar{w} dominating $\max(u_1, u_2)$. This solution satisfies*

$$(3.28) \quad \max(\mu_1, \mu_2) \leq \text{tr}(\bar{w}) \leq \mu_1 + \mu_2.$$

There exists a nonnegative maximal solution \underline{w} dominated by $\min(u_1, u_2)$. This solution satisfies

$$(3.29) \quad \text{tr}(\underline{w}) \leq \min(u_1, u_2).$$

If, in addition, $\text{supp } \mu_1 \cap \text{supp } \mu_2 = \emptyset$ then $\text{tr}(\bar{w}) = \mu_1 + \mu_2$. In this case, there exists no positive solution dominated by $\min(u_1, u_2)$.

Proof. For every $\delta \in (0, \delta_0)$, denote by $w := \bar{w}_\delta$ the solution of

$$(3.30) \quad \begin{cases} -\Delta w + H \circ w = 0 & \text{in } D_\delta \\ w = \max(u_1, u_2) & \text{on } \Sigma_\delta. \end{cases}$$

By the comparison principle, $\max(u_1, u_2) \leq \bar{w}_\delta \leq \mathbb{P}^\Omega[\mu_1 + \mu_2]$ in D_δ . Consequently, the sequence $\{\bar{w}_\delta\}$ is increasing and bounded from above by $\mathbb{P}^\Omega[\mu_1 + \mu_2]$. Therefore, $\bar{w} := \lim_{\delta \rightarrow 0} \bar{w}_\delta$ is a solution of (1.1) satisfying

$$\max(u_1, u_2) \leq \bar{w} \leq \mathbb{P}^\Omega[\mu_1 + \mu_2]$$

in Ω . By Theorem 3.7 \bar{w} admits a boundary trace and (3.28) holds.

If w is a solution of (1.1) dominating $\max\{u_1, u_2\}$ then by the comparison principle, $w \geq \bar{w}_\delta$ for every $\delta > 0$. It follows that $w \geq \bar{w}$ and therefore \bar{w} is a minimal solution dominating $\max(u_1, u_2)$.

For every $\delta \in (0, \delta_0)$, denote by $w := \underline{w}_\delta$ the solution of

$$(3.31) \quad \begin{cases} -\Delta w + H \circ w = 0 & \text{in } D_\delta \\ w = \min(u_1, u_2) & \text{on } \Sigma_\delta. \end{cases}$$

The sequence $\{\underline{w}_\delta\}$ is decreasing and converges, as $\delta \rightarrow 0$, to a function \underline{w} which is a solution of (1.1) such that $0 \leq \underline{w} \leq \min(u_1, u_2)$ in Ω . As above, one can show that \underline{w} is the maximal solution dominated by $\min(u_1, u_2)$ and (3.29) holds.

If $\text{supp } \mu_1 \cap \text{supp } \mu_2 = \emptyset$ then $\text{tr}(\max(u_1, u_2)) = \mu_1 + \mu_2$ and $\text{tr}(\min(u_1, u_2)) = 0$. Therefore $\text{tr}(\overline{w}) = \mu_1 + \mu_2$ and $\text{tr}(\underline{w}) = 0$. Thus $\underline{w} \equiv 0$. \square

4. ISOLATED BOUNDARY SINGULARITIES

If u is a solution of (1.1) in Ω with an isolated singularity at a point $A \in \partial\Omega$, we shall assume that the set of coordinates is chosen so that A is the origin.

4.1. Weakly singular solutions. We start with some a-priori estimates regarding solutions with an isolated singularity.

Lemma 4.1. *Assume $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ is a nonnegative solution of (1.1) in Ω vanishing on $\partial\Omega \setminus \{0\}$.*

(i) *Assume H satisfies (1.7). Then*

$$(4.1) \quad u(x) \leq \Lambda_1 |x|^{-\beta_1} \quad \forall x \in \Omega,$$

$$(4.2) \quad |\nabla u(x)| \leq \Lambda_3 |x|^{-\beta_1-1} \quad \forall x \in \Omega,$$

$$(4.3) \quad u(x) \leq \tilde{\Lambda}_3 \rho(x) |x|^{-\beta_1-1} \quad \forall x \in \Omega$$

where Λ_1 is defined in (2.3), $\Lambda_3 = \Lambda_3(N, p, q, \Omega)$ and $\tilde{\Lambda}_3 = \tilde{\Lambda}_3(N, p, q, \Omega)$.

(ii) *Assume H satisfies (1.8). Then*

$$(4.4) \quad u(x) \leq \Lambda_2 |x|^{-\beta_2} \quad \forall x \in \Omega,$$

$$(4.5) \quad |\nabla u(x)| \leq \Lambda_4 |x|^{-\beta_2-1} \quad \forall x \in \Omega,$$

$$(4.6) \quad u(x) \leq \tilde{\Lambda}_4 \rho(x) |x|^{-\beta_2-1} \quad \forall x \in \Omega$$

where $\Lambda_2 = \Lambda_2(p, q)$, $\Lambda_4 = \Lambda_4(N, p, q, \Omega)$ and $\tilde{\Lambda}_4 = \tilde{\Lambda}_4(N, p, q, \Omega)$.

Proof. We deal only with the case where H satisfies (1.7) since the case H satisfies (1.8) can be treated in a similar way. For $\epsilon > 0$, we set

$$P_\epsilon(r) = \begin{cases} 0 & \text{if } r \leq \epsilon \\ \frac{-r^4}{2\epsilon^3} + \frac{3r^3}{\epsilon^2} - \frac{6r^2}{\epsilon} + 5r - \frac{3\epsilon}{2} & \text{if } \epsilon < r < 2\epsilon \\ r - \frac{3\epsilon}{2} & \text{if } r \geq 2\epsilon \end{cases}$$

and let u_ϵ be the extension of $P_\epsilon(u)$ by zero outside Ω . There exists R_0 such that $\Omega \subset B_{R_0}$. Since $0 \leq P'_\epsilon(r) \leq 1$ and P_ϵ is convex, $u_\epsilon \in C^2(\mathbb{R}^N \setminus \{0\})$ and it satisfies $-\Delta u_\epsilon + u_\epsilon^p |\nabla u_\epsilon|^q \leq 0$. Furthermore u_ϵ vanishes in $B_{R_0}^c$. For $\delta > 0$, we set

$$U_\delta(x) = \Lambda_1(|x| - \delta)^{-\beta_1} \quad \forall x \in \mathbb{R}^N \setminus B_\delta,$$

then $-\Delta U_\delta + U_\delta^p |\nabla U_\delta|^q \geq 0$ in $B_{R_0} \setminus B_\delta$. Since u_ϵ vanishes on ∂B_{R_0} and is finite on ∂B_δ , by the comparison principle, $u_\epsilon \leq U_\delta$ in $B_{R_0} \setminus \overline{B}_\delta$. Letting successively $\delta \rightarrow 0$ and $\epsilon \rightarrow 0$ yields to (4.1).

For $\ell > 0$, define $T_\ell^1[u](x) = \ell^{\beta_1} u(\ell x)$, $x \in \Omega^\ell := \ell^{-1}\Omega$. If $x_0 \in \Omega$, we set $r_0 = |x_0|$ and $u_{r_0}(x) = T_{r_0}^1[u](x)$. Then u_{r_0} satisfies (1.1) in $\Omega^{r_0} = r_0^{-1}\Omega$. By (4.1),

$$\max\{|u_{r_0}(x)| : (B_{\frac{3}{2}} \setminus B_{\frac{1}{2}}) \cap \Omega^{r_0}\} \leq 2^{\beta_1} \Lambda_1.$$

By regularity results [10, Theorem 1], there exists $\Lambda_3 = \Lambda_3(N, \Omega, p, q)$ such that

$$\max\{|\nabla u_{r_0}(x)| : (B_{\frac{5}{4}} \setminus B_{\frac{3}{4}}) \cap \Omega^{r_0}\} \leq \Lambda_3.$$

In particular, $|\nabla u_{r_0}(x)| \leq \Lambda_3$ with $|x| = 1$. Hence $|\nabla u(x_0)| \leq \Lambda_3 |x_0|^{-\beta_1-1}$.

Finally, (4.3) follows from (4.1) and (4.2). \square

An uniqueness result for (1.9) can be obtained if μ is a bounded measure concentrated at a point on $\partial\Omega$. We assume that the point is the origin.

Theorem 4.2. *Assume either H satisfies (1.7) with $0 < N(p+q-1) < p+1$ or H satisfies (1.8) with $m_{p,q} < p_c$ where p_c and $m_{p,q}$ are given in (1.3) and (1.11) respectively. Then for every $k > 0$, there exists a unique solution, denoted by $u_{k,0}^\Omega$, of the problem*

$$(4.7) \quad \begin{cases} -\Delta u + H \circ u &= 0 & \text{in } \Omega \\ u &= k\delta_0 & \text{on } \partial\Omega. \end{cases}$$

Moreover,

$$(4.8) \quad u_{k,0}^\Omega(x) = k(1 + o(1))P^\Omega(x, 0) \quad \text{as } x \rightarrow 0.$$

Consequently the mapping $k \mapsto u_{k,0}^\Omega$ is increasing.

The existence of a solution to (4.7) is guaranteed by Theorem A. The proof of uniqueness is based on the following lemma.

Lemma 4.3. *Under the assumption of Theorem 4.2, let u be a solution to (4.7). Then*

$$(4.9) \quad \mathbb{G}^\Omega[H \circ u](x) = o(P^\Omega(x, 0)) \quad \text{as } x \rightarrow 0.$$

Proof. We prove (4.9) in the case H satisfies (1.7). The case H satisfies (1.8) can be treated in a similar way.

Since u is a solution of (4.7), it follows from the maximum principle that

$$u(x) \leq kP^\Omega(x, 0) \leq kc_N|x|^{1-N} \quad \forall x \in \Omega$$

where c_N is a positive constant depending on N and Ω . By adapting argument in the proof of Lemma 4.1, we obtain

$$(4.10) \quad |\nabla u(x)| \leq \Lambda_5 k |x|^{-N} \quad \forall x \in \Omega$$

where Λ_5 is a positive constant depending on N, p, q, Ω . Consequently,

$$(4.11) \quad \mathbb{G}^\Omega[H \circ u](x) \leq c_8 \int_\Omega G^\Omega(x, y) |y|^{-(N-1)p-Nq} dy \quad \forall x \in \Omega.$$

By [21], there exists $c_9 = c_9(N, \Omega)$ such that, for $\varepsilon_0 \in (0, 1)$,

$$(4.12) \quad G^\Omega(x, y) \leq c_9 \rho(x) \rho(y)^{1-\varepsilon_0} |x - y|^{\varepsilon_0 - N} \quad \forall x, y \in \Omega, x \neq y,$$

This, joint with (4.11), implies

$$(4.13) \quad \mathbb{G}^\Omega[H \circ u](x) \leq c_{10} |x|^N P^\Omega(x, 0) \int_{\mathbb{R}^N} |x - y|^{\varepsilon_0 - N} |y|^{1-(N-1)p-Nq-\varepsilon_0} dy.$$

We fix ε_0 such that $0 < \varepsilon_0 < \min\{1, N+1-(N-1)p-Nq\}$. By the following identity (see [11]),

$$(4.14) \quad \int_{\mathbb{R}^N} |x - y|^{\varepsilon_0 - N} |y|^{1-(N-1)p-Nq-\varepsilon_0} dy = c_{11} |x|^{N+1-(N-1)p-Nq}$$

where $c_{11} = c_{11}(N, \varepsilon_0)$, we obtain

$$\mathbb{G}^\Omega[H \circ u](x) \leq \alpha_1 c_{10} c_{11} |x|^{N+1-(N-1)p-Nq} P^\Omega(x, 0).$$

Since $N+1-(N-1)p-Nq > 0$, by letting $x \rightarrow 0$, we obtain (4.9). \square

Proof of Theorem 4.2. Let u_1 and u_2 be two solutions of (4.7) then $u_i(x) = k P^\Omega(x, 0) - \mathbb{G}^\Omega[H \circ u_i](x)$. From (4.9), we obtain

$$(4.15) \quad u_i(x) = k(1 + o(1)) P^\Omega(x, 0) \quad \text{as } x \rightarrow 0.$$

By the comparison principle, we deduce $u_1 = u_2$. The monotonicity of $k \mapsto u_{k,0}^\Omega$ follows from (4.8) and the comparison principle. \square

Since $\partial\Omega$ is of class C^2 , there exists $r_0 > 0$ such that for every $y \in \partial\Omega$, $B_{r_0}(y - r_0 \mathbf{n}_y) \subset \Omega$ where \mathbf{n}_y is the outward normal unit vector at y .

Lemma 4.4. Assume H satisfies either (1.7) with $p+q > 1$ and $q < 2$ or H satisfies (1.8) with $p > 1$, $1 < q < 2$. Let $u \in C^2(\Omega)$ be a positive solution of (1.1). Let $y \in \partial\Omega$ be such that u is continuous at y and $u(y) = 0$. Then

$$(4.16) \quad \liminf_{x \rightarrow y} \frac{u(x)}{|x - y|} > 0 \quad n.t.$$

Proof. We only deal with the case H satisfies (1.7) since the case H satisfies (1.8) can be treated in a similar way. Put $z = y - r_0 \mathbf{n}_y$ and set

$$v(x) = e^{-\alpha|x-z|^2} - e^{-\alpha r_0^2} \quad x \in B_{r_0}(z) \setminus B_{r_0/2}(z)$$

where $\alpha > 0$ will be determined latter on. Then, in $B_{r_0}(z) \setminus B_{r_0/2}(z)$,

$$\begin{aligned} -\Delta v + v^p |\nabla v|^q &= 2\alpha N e^{-\alpha r^2} - 4\alpha^2 r^2 e^{-\alpha r^2} + (e^{-\alpha r^2} - e^{-\alpha r_0^2})^p (2\alpha)^q r^q e^{-\alpha q r^2} \\ &\leq 2\alpha(N - 2\alpha r^2 + 2^{q-1} \alpha^{\alpha-1} r^q) e^{-\alpha r^2} \\ &\leq \alpha(2N - \alpha r_0^2 + 2^q \alpha^{q-1} r_0^q) e^{-\alpha r^2}. \end{aligned}$$

Since $q < 2$, one can choose α large enough such that the last expression is negative. Consequently, v is a subsolution of (1.7). As u is positive on $\partial B_{r_0/2}(z)$, one can

choose ε small such that $u > \varepsilon v$ on $\partial B_{r_0/2}(z)$. Obviously $u \geq 0 = \varepsilon v$ on $\partial B_{r_0}(z)$. By the comparison principle, $u \geq \varepsilon v$ in $B_{r_0}(z) \setminus B_{r_0/2}(z)$. It follows that

$$\liminf_{x \rightarrow y} \frac{u(x)}{|x - y|} \geq -\varepsilon \frac{\partial v}{\partial \mathbf{n}}(y) = 2\varepsilon \alpha r_0 e^{-\alpha r_0^2} > 0.$$

□

Proposition 4.5. *Under the assumption of Theorem 4.2, for every $k > 0$, there exists a positive constant d_k depending on N, p, q, k and Ω such that*

$$(4.17) \quad d_k P^\Omega(x, 0) < u_{k,0}^\Omega(x) < k P^\Omega(x, 0) \quad \forall x \in \Omega.$$

Proof. The second inequality follows straightforward from the comparison principle. In order to prove the first inequality, put $\mathcal{A} = \{d > 0 : d P^\Omega(\cdot, 0) < u_{k,0}^\Omega \text{ in } \Omega\}$. Suppose by contradiction that $\mathcal{A} = \emptyset$. Then for each $n \in \mathbb{N}$, there exists a point $x_n \in \Omega$ such that

$$(4.18) \quad n u_{k,0}^\Omega(x_n) \leq P^\Omega(x_n, 0).$$

We may assume that $\{x_n\}$ converges to a point $x^* \in \overline{\Omega}$. We deduce from (4.18) that $x^* \notin \Omega$. Thus $x^* \in \partial\Omega$. By Theorem 4.2, $x^* \in \partial\Omega \setminus B_\epsilon(0)$ for some $\epsilon > 0$. Denote by σ_{x_n} the projection of x_n on $\partial\Omega$. It follows from (4.18) that

$$\frac{u_{k,0}^\Omega(\sigma_{x_n}) - u_{k,0}^\Omega(x_n)}{\rho(x_n)} \geq \frac{1}{n} \frac{P^\Omega(\sigma_{x_n}, 0) - P^\Omega(x_n, 0)}{\rho(x_n)}.$$

By letting $n \rightarrow \infty$, we obtain

$$\frac{\partial u_{k,0}^\Omega}{\partial \mathbf{n}}(0) = 0$$

which contradicts (4.16). Thus $\mathcal{A} \neq \emptyset$. Put $d_k = \max \mathcal{A}$. By combining (4.1) and boundary Harnack inequality, we deduce that d_k depends on N, p, q, k and Ω . □

Proof of Theorem C. The proof follows from Theorem 4.2 and Proposition 4.5. □

The next result gives the existence and uniqueness of the weakly singular solution in the case that Ω is an unbounded domain.

Theorem 4.6. *Under the assumption of Theorem 4.2, let either $\Omega = \mathbb{R}_+^N := [x_N > 0]$ or $\partial\Omega$ be compact with $0 \in \partial\Omega$ (Ω is possibly unbounded). Then there exists a unique solution $u_{k,0}^\Omega$ to (4.7).*

Proof. If $\partial\Omega$ is compact, for $n \in \mathbb{N}$ large enough, $\partial\Omega \subset B_n(0)$. We set $\Omega_n = \Omega \cap B_n(0)$ and denote by $u_{k,0}^{\Omega_n}$ the unique solution of

$$(4.19) \quad \begin{cases} -\Delta u + H \circ u = 0 & \text{in } \Omega_n \\ u = k\delta_0 & \text{on } \partial\Omega_n. \end{cases}$$

Then

$$(4.20) \quad u_{k,0}^{\Omega_n}(x) \leq k P^{\Omega_n}(x, 0) \quad \forall x \in \Omega_n$$

and

$$(4.21) \quad u_{k,0}^{\Omega_n}(x) = k(1 + o(1))P^{\Omega_n}(x, 0) \quad \text{as } x \rightarrow 0.$$

Thus $\{u_{k,0}^{\Omega_n}\}$ increase to a function u^* which satisfies

$$(4.22) \quad u^*(x) \leq kP^{\Omega}(x, 0) \quad \forall x \in \Omega.$$

By regularity theory, $\{u_{k,0}^{\Omega_n}\}_n$ converges in $C_{loc}^1(\overline{\Omega} \setminus \{0\})$ when $n \rightarrow \infty$, and thus $u^* \in C(\overline{\Omega} \setminus \{0\})$ is a positive solution of (1.1) in Ω vanishing on $\partial\Omega \setminus \{0\}$. Estimate (4.22) implies that the boundary trace of u^* is a Dirac measure at 0, which is in fact $k\delta_0$ due to (4.21). Uniqueness follows from the comparison principle. \square

4.2. Strongly singular solutions. In this section, we establish existence and uniqueness of strongly singular solutions at a boundary point. We assume that the point is the origin.

Lemma 4.7. *Under the assumption of Theorem C, if v is a positive solution of (1.1) and $y \in \mathcal{S}(v)$ then $v \geq u_{\infty,y}^{\Omega}$.*

Proof. We can suppose that y is the origin. Since $0 \in \mathcal{S}(v)$, by Theorem 3.9 for every $n \in \mathbb{N}_*$,

$$\limsup_{\beta \rightarrow 0} \int_{B_{\frac{1}{n}}(0) \cap \Sigma_{\delta}} v dS = \infty.$$

Consequently, there exists a sequence $\{\delta_{n,m}\}_{m \in \mathbb{N}}$ tending to zero as $m \rightarrow \infty$ such that

$$\lim_{m \rightarrow \infty} \int_{\Sigma_{\delta_{n,m}} \cap B_{\frac{1}{n}}(0)} v dS = \infty.$$

Then, for any $k > 0$, there exists $m_k := m_{n,k} \in \mathbb{N}$ such that

$$(4.23) \quad m \geq m_k \implies \int_{\Sigma_{\delta_{n,m_k}} \cap B_{\frac{1}{n}}(0)} v dS \geq k$$

and $m_k \rightarrow \infty$ when $n \rightarrow \infty$. In particular there exists $t := t(n, k) > 0$ such that

$$(4.24) \quad \int_{\Sigma_{\delta_{n,m_k}} \cap B_{\frac{1}{n}}(0)} \inf\{v, t\} dS = k.$$

By the comparison principle v is bounded from below in $D_{\delta_{n,m_k}}$ by the solution $w := w_{\delta_{n,m_k}}$ of

$$(4.25) \quad \begin{cases} -\Delta w + H \circ w = 0 & \text{in } D_{\delta_{n,m_k}} \\ w = \inf\{v, t\} & \text{on } \Sigma_{\delta_{n,m_k}}. \end{cases}$$

When $n \rightarrow \infty$, $\inf\{v, t(n, k)\} dS$ converges weakly to $k\delta_0$. By Corollary 3.1 there exists a subsequence, still denoted by $\{w_{\delta_{n,m_k}}\}_n$, such that $w_{\delta_{n,m_k}} \rightarrow u_{k,0}^{\Omega}$ when

$n \rightarrow \infty$ where $u_{k,0}^\Omega$ is the unique solution of (4.7) and consequently $v \geq u_{k,0}^\Omega$ in Ω . Therefore $v \geq u_{\infty,0}^\Omega$. \square

Proof of Theorem D. By Theorem C and Lemma 4.1, the sequence $\{u_{k,0}^\Omega\}$ is nondecreasing and bounded from above by either $\tilde{\Lambda}_3\rho(x)|x|^{-\beta_1-1}$ if H satisfies (1.7) or $\tilde{\Lambda}_4\rho(x)|x|^{-\beta_2-1}$ if H satisfies (1.8). Therefore $\{u_{k,0}^\Omega\}$ converges to a function $u_{\infty,0}^\Omega$. By regularity theory, $u_{\infty,0}^\Omega$ is a solution of (1.1) vanishing on $\partial\Omega \setminus \{0\}$. Moreover, since $u_{\infty,0}^\Omega \geq u_{k,0}^\Omega$ for every $k > 0$, $\mathcal{S}(u_{\infty,0}^\Omega) = \{0\}$ and therefore $u_{\infty,0}^\Omega \in \mathcal{U}_0^\Omega$. If $v \in \mathcal{U}_0^\Omega$ then by Lemma 4.7, $v \geq u_{\infty,0}^\Omega$. Thus $u_{\infty,0}^\Omega$ is the minimal element of \mathcal{U}_0^Ω . \square

For any $\ell > 0$ and any solution of (1.1), define

$$(4.26) \quad \Omega^\ell = \ell^{-1}\Omega, \quad T_\ell^1[u](x) = \ell^{\beta_1}u(\ell x), \quad T_\ell^2[u](x) = \ell^{\beta_2}u(\ell x) \quad \forall x \in \Omega^\ell.$$

Proposition 4.8. *Let $v \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ be a nonnegative solution of (1.1) vanishing on $\partial\Omega \setminus \{0\}$.*

(1) *Assume H satisfies (1.7). For each $\ell > 0$, put $v_\ell(x) = T_\ell^1[v](x)$. Then, up to a subsequence, $\{v_\ell\}$ converges in $C_{loc}^1(\mathbb{R}_+^N \setminus \{0\})$, as $\ell \rightarrow 0$, to a solution of*

$$(4.27) \quad -\Delta u + u^p |\nabla u|^q = 0 \text{ in } \mathbb{R}_+^N, \quad u = 0 \text{ on } \partial\mathbb{R}_+^N \setminus \{0\}.$$

(2) *Assume H satisfies (1.8). For each ℓ , put $v_\ell(x) = T_\ell^2[v](x)$.*

(i) *If $p = \frac{q}{2-q}$ then, up to a subsequence, $\{v_\ell\}$ converges in $C_{loc}^1(\overline{\mathbb{R}_+^N} \setminus \{0\})$, as $\ell \rightarrow 0$, to a solution of*

$$(4.28) \quad -\Delta u + u^p + |\nabla u|^q = 0 \text{ in } \mathbb{R}_+^N, \quad u = 0 \text{ on } \partial\mathbb{R}_+^N \setminus \{0\}.$$

(ii) *If $p > \frac{q}{2-q}$ then, up to a subsequence, $\{v_\ell\}$ converges in $C_{loc}^1(\overline{\mathbb{R}_+^N} \setminus \{0\})$, as $\ell \rightarrow 0$, to a solution of*

$$(4.29) \quad -\Delta u + u^p = 0 \text{ in } \mathbb{R}_+^N, \quad u = 0 \text{ on } \partial\mathbb{R}_+^N \setminus \{0\}.$$

(iii) *If $p < \frac{q}{2-q}$ then, up to a subsequence, $\{v_\ell\}$ converges in $C_{loc}^1(\overline{\mathbb{R}_+^N} \setminus \{0\})$, as $\ell \rightarrow 0$, to a solution of*

$$(4.30) \quad -\Delta u + |\nabla u|^q = 0 \text{ in } \mathbb{R}_+^N, \quad u = 0 \text{ on } \partial\mathbb{R}_+^N \setminus \{0\}.$$

Proof. We first notice that if H satisfies either (1.7) or (1.8) with $p = \frac{q}{2-q}$ then v_ℓ is a solution of (1.1) in Ω^ℓ which vanishes on $\partial\Omega^\ell \setminus \{0\}$. If H satisfies (1.8) with $p > \frac{q}{2-q}$ then v_ℓ satisfies

$$(4.31) \quad -\Delta v_\ell + v_\ell^p + \ell^{\frac{p(2-q)-q}{p-1}} |\nabla v_\ell|^q = 0 \text{ in } \Omega^\ell, \quad v_\ell = 0 \text{ on } \partial\Omega^\ell \setminus \{0\}.$$

If H satisfies (1.8) with $p < \frac{q}{2-q}$ then v_ℓ satisfies

$$(4.32) \quad -\Delta v_\ell + \ell^{\frac{q-(2-q)p}{q-1}} v_\ell^p + |\nabla v_\ell|^q = 0 \text{ in } \Omega^\ell, \quad v_\ell = 0 \text{ on } \partial\Omega^\ell \setminus \{0\}.$$

Next, it follows from Lemma 4.1 and [10, Theorem 1] that for every $R > 1$ there exist positive numbers $M = M(N, p, q, R)$ and $\gamma = \gamma(N, p, q) \in (0, 1)$ such that

$$(4.33) \quad \sup\{|v_\ell(x)| + |\nabla v_\ell(x)| : x \in \Gamma_{R^{-1}, R} \cap \Omega^\ell\} + \sup\left\{\frac{|\nabla v_\ell(x) - \nabla v_\ell(y)|}{|x - y|^\gamma} : x, y \in \Gamma_{R^{-1}, R} \cap \Omega^\ell\right\} \leq M$$

where $\Gamma_{t_1, t_2} := B_{t_2}(0) \setminus B_{t_1}(0)$ with $0 < t_1 < t_2$. Thus there exists a sequence $\{\ell_n\}$ and a function $v^* \in C^1(\overline{\mathbb{R}_+^N} \setminus \{0\})$ such that $\{v_{\ell_n}\}$ converges to v^* in $C_{loc}^1(\overline{\mathbb{R}_+^N} \setminus \{0\})$ which is a solution of

$$(4.34) \quad \begin{cases} -\Delta v + H \circ v = 0 & \text{in } \mathbb{R}_+^N \\ v = 0 & \text{in } \partial\mathbb{R}_+^N \setminus \{0\} \end{cases}$$

Moreover,

$$(4.35) \quad \lim_{n \rightarrow \infty} (\sup\{|(v_{\ell_n} - v^*)(x)| + |\nabla(v_{\ell_n} - v^*)(x)| : x \in \Gamma_{R^{-1}, R} \cap \Omega^{\ell_n}\}) = 0.$$

□

Proposition 4.9. *Let $v = u_{\infty, 0}^\Omega$ and $\{v_\ell\}$ be defined as in Proposition 4.8. Then, up to a subsequence, $\{v_\ell\}$ converges to a strongly singular solution of*

$$(4.36) \quad \begin{cases} (4.27) \text{ if } H \text{ satisfies (1.7)} \\ (4.28) \text{ if } H \text{ satisfies (1.8) with } p = \frac{q}{2-q} \\ (4.29) \text{ if } H \text{ satisfies (1.8) with } p > \frac{q}{2-q} \\ (4.30) \text{ if } H \text{ satisfies (1.8) with } p < \frac{q}{2-q} \end{cases}$$

Moreover, if H satisfies (1.8) and $p \neq \frac{q}{2-q}$ then the whole sequence $\{v_\ell\}$ converges to the strongly singular solution of (4.29) if $p > \frac{q}{2-q}$ or it converges to the strongly singular solution of (4.30) if $p < \frac{q}{2-q}$.

Proof. Since $v_\ell \geq u_{k, 0}^{\Omega_\ell}$ for every $\ell > 0$ and $k > 0$, $v^* \geq u_{k, 0}^{\mathbb{R}_+^N}$ for every $k > 0$ (v^* is given in the proof of Proposition 4.8). Therefore v^* is a strongly singular solution.

If H satisfies (1.8) with $p \neq \frac{q}{2-q}$ then by the uniqueness of the strongly singular solution of (4.29) and (4.30) (see [20] and [22]), we get the conclusion. □

Denote by \mathcal{E}_i ($i = 1, \dots, 4$) the set of positive solutions in $C^2(S_+^{N-1})$ of

$$(4.37) \quad -\Delta' \omega + F_i(\omega, \nabla' \omega) = 0 \text{ in } S_+^{N-1}, \quad \omega = 0 \text{ on } \partial S_+^{N-1}$$

where F_i is as in (1.20).

We next study the structure of \mathcal{E}_i .

Theorem 4.10. (i) Assume either H satisfies (1.7) with $N(p+q-1) \geq p+1$ or H satisfies (1.8) with $m_{p,q} \geq p_c$. Then $\mathcal{E}_i = \emptyset$ where

$$(4.38) \quad i = \begin{cases} 1 & \text{if } H \text{ satisfies (1.7)} \\ 2 & \text{if } H \text{ satisfies (1.8) with } p = \frac{q}{2-q} \\ 3 & \text{if } H \text{ satisfies (1.8) with } p > \frac{q}{2-q} \\ 4 & \text{if } H \text{ satisfies (1.8) with } p < \frac{q}{2-q}. \end{cases}$$

(ii) Assume either H satisfies (1.7) with $0 < N(p+q-1) < p+1$ or H satisfies (1.8) with $m_{p,q} < p_c$. Then $\mathcal{E}_i \neq \emptyset$ with i as in (4.38).

Proof. Notice that if H satisfies (1.8) with $p \neq \frac{q}{2-q}$, statements (i) and (ii) have been proved in [20] and [22]. More precisely, if $m_{p,q} < p_c$ and $p > \frac{q}{2-q}$ then there exists a unique element ω_3 of \mathcal{E}_3 , while if $m_{p,q} < p_c$ and $p < \frac{q}{2-q}$ then there exists a unique element ω_4 of \mathcal{E}_4 .

So we are left with the case when H satisfies either (1.7) or H satisfies (1.8) with $p = \frac{q}{2-q}$ and we only give the proof for the case H satisfies (1.7).

(i) Denote by φ_1 the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$, normalized such that $\max_{S_+^{N-1}} \varphi_1 = 1$, with corresponding eigenvalue $\lambda_1 = N-1$. Multiplying (4.37) by φ_1 and integrating over S_+^{N-1} , we get

$$[N-1-\beta_1(\beta_1+2-N)] \int_{S_+^{N-1}} \omega \varphi_1 dS(\sigma) + \int_{S_+^{N-1}} \omega^p (\beta_1^2 \omega^2 + |\nabla' \omega|^2)^{\frac{q}{2}} \varphi_1 dS(\sigma) = 0.$$

Therefore if $N-1 \geq \beta_1(\beta_1+2-N)$, namely $N(p+q-1) \geq p+1$, then there exists no positive solution of (4.37).

(ii) The proof is based on the construction of a subsolution and a supersolution to (4.37). By a simple computation, we can prove that $\underline{\omega} := \theta_1 \varphi_1^{\theta_2}$ is a positive subsolution of (4.37) with $\theta_1 > 0$ small and $1 < \theta_2 < \frac{\beta_1(\beta_1+2-N)}{N-1}$. Next, it is easy to see that $\overline{\omega} = \theta_3$, with $\theta_3 > 0$ large enough, is a supersolution of (4.37) and $\overline{\omega} > \underline{\omega}$ in $\overline{S_+^{N-1}}$. Therefore by [13] there exists a solution $\omega_1 \in W^{2,m}(S_+^{N-1})$ (for any $m > N$) to (4.37) such that $0 < \underline{\omega} \leq \omega_1 \leq \overline{\omega}$ in S_+^{N-1} . By regularity theory, $\omega_1 \in C^2(\overline{S_+^{N-1}})$.

Similarly, we can show that if $m_{p,q} < p_c$ and $p = \frac{q}{2-q}$ then there exists a function $\omega_2 \in \mathcal{E}_2$. \square

We next show that ω_i ($i = 1, 2$) is the unique element of \mathcal{E}_i .

Theorem 4.11. (i) If H satisfies (1.7) with $N(p+q-1) < p+1$ and $p \geq 1$ then $\mathcal{E}_1 = \{\omega_1\}$.

(ii) If H satisfies (1.8) with $m_{p,q} < p_c$ and $p = \frac{q}{2-q}$ then $\mathcal{E}_2 = \{\omega_2\}$.

Proof. We give below only the proof of statement (i); the statement (ii) can be treated in a similar way. Suppose that ω_1 and ω'_1 are two positive different solutions

of (4.37). Up to exchanging the role of ω_1 and ω'_1 , we may assume $\max_{S_+^{N-1}} \omega'_1 \geq \max_{S_+^{N-1}} \omega_1$ and

$$\tau_0 := \inf\{\tau > 1 : \tau\omega_1 > \omega'_1 \text{ in } S_+^{N-1}\} > 1.$$

Set $\omega_{1,\tau_0} := \tau_0\omega_1$, then ω_{1,τ_0} is a positive supersolution to problem (4.37). Put $\tilde{\omega} = \omega_{1,\tau_0} - \omega'_1 \geq 0$. If there exists $\sigma_0 \in S_+^{N-1}$ such that $\omega_{1,\tau_0}(\sigma_0) = \omega'_1(\sigma_0) > 0$ and $\nabla'\omega_{1,\tau_0}(\sigma_0) = \nabla'\omega'_1(\sigma_0)$ then $\tilde{\omega}(\sigma_0) = 0$ and $\nabla'\tilde{\omega}(\sigma_0) = 0$. This contradicts the strong maximum principle (see [7]). If $\omega_{1,\tau_0} > \omega'_1$ in S_+^{N-1} and there exists $\sigma_0 \in \partial S_+^{N-1}$ such that $\frac{\partial\omega_{1,\tau_0}}{\partial\mathbf{n}}(\sigma_0) = \frac{\partial\omega'_1}{\partial\mathbf{n}}(\sigma_0)$ then $\tilde{\omega} > 0$ and $\frac{\partial\tilde{\omega}}{\partial\mathbf{n}}(\sigma_0) = 0$. This contradicts the Hopf lemma (see [7]). \square

Let T_ℓ^1 and T_ℓ^2 be as in (4.26). If u is a solution of (1.1) in Ω with H as in (1.7) (resp. H as in (1.8) and $p = \frac{q}{2-q}$) then $T_\ell^1[u]$ (resp. $T_\ell^2[u]$) is a solution of (1.1) in $\Omega^\ell = \ell^{-1}\Omega$. If $\Omega = \Omega^\ell$ and $u = T_\ell^1[u]$ (resp. $u = T_\ell^2[u]$) for every $\ell > 0$ we say that u is a *self-similar solution*.

For $x \in \mathbb{R}^N \setminus \{0\}$, put $r = |x|$ and $\sigma = \frac{x}{r}$.

Proposition 4.12. (i) If H satisfies (1.7) with $N(p+q-1) < p+1$ and $p \geq 1$ then

$$(4.39) \quad r^{\beta_1} u_{\infty,0}^\Omega(x) \rightarrow \omega_1(\sigma) \quad \text{as } r \rightarrow 0, \ x \in \Omega, \ \sigma \in S_+^{N-1}$$

locally uniformly on S_+^{N-1} .

(ii) If H satisfies (1.8) with $m_{p,q} < p_c$ then

$$(4.40) \quad r^{\beta_2} u_{\infty,0}^\Omega(x) \rightarrow \omega_i(\sigma) \quad \text{as } r \rightarrow 0, \ x \in \Omega, \ \sigma \in S_+^{N-1}$$

locally uniformly on S_+^{N-1} where $i \geq 2$ is as in (4.38).

Proof. Case 1: H satisfies (1.7). Since the proof is close to the one of [22, Proposition 3.22], we present only the main ideas.

We first note that $T_\ell^1[u_{\infty,0}^{\mathbb{R}_+^N}] = u_{\infty,0}^{\mathbb{R}_+^N}$ for every $\ell > 0$. Hence $u_{\infty,0}^{\mathbb{R}_+^N}$ is self-similar and satisfies (4.39) with Ω replaced by \mathbb{R}_+^N .

Next, let B and B' are two open balls tangent to $\partial\Omega$ at 0 such that $B \subset \Omega \subset G := (B')^c$. Then

$$(4.41) \quad u_{\infty,0}^{B^{\ell'}} \leq u_{\infty,0}^{B^\ell} \leq u_{\infty,0}^{\mathbb{R}_+^N} \leq u_{\infty,0}^{G^\ell} \leq u_{\infty,0}^{G^{\ell''}} \quad \forall 0 < \ell \leq \ell', \ell'' \leq 1.$$

Notice that $u_{\infty,0}^{B^\ell} \uparrow \underline{u}_{\infty,0}^{\mathbb{R}_+^N}$ and $u_{\infty,0}^{G^\ell} \downarrow \overline{u}_{\infty,0}^{\mathbb{R}_+^N}$ when $\ell \rightarrow 0$ where $\underline{u}_{\infty,0}^{\mathbb{R}_+^N}$ and $\overline{u}_{\infty,0}^{\mathbb{R}_+^N}$ are positive solutions of (1.1) in \mathbb{R}_+^N , continuous in $\overline{\mathbb{R}_+^N} \setminus \{0\}$ and vanishing on $\partial\mathbb{R}_+^N \setminus \{0\}$. By letting $\ell \rightarrow 0$ in (4.41), we obtain

$$(4.42) \quad u_{\infty,0}^{B^\ell} \leq \underline{u}_{\infty,0}^{\mathbb{R}_+^N} \leq u_{\infty,0}^{\mathbb{R}_+^N} \leq \overline{u}_{\infty,0}^{\mathbb{R}_+^N} \leq u_{\infty,0}^{G^\ell} \quad \forall 0 < \ell \leq 1.$$

Furthermore there also holds for $\ell, \ell' > 0$,

$$(4.43) \quad T_{\ell'\ell}^1[u_{\infty,0}^B] = T_{\ell'}^1[T_\ell^1[u_{\infty,0}^B]] = u_{\infty,0}^{B^{\ell\ell'}} \text{ and } T_{\ell'\ell}^1[u_{\infty,0}^G] = T_{\ell'}^1[T_\ell^1[u_{\infty,0}^G]] = u_{\infty,0}^{G^{\ell\ell'}}.$$

Letting $\ell \rightarrow 0$ in (4.43) yields

$$(4.44) \quad \underline{u}_{\infty,0}^{\mathbb{R}_+^N} = T_{\ell'}^1[\underline{u}_{\infty,0}^{\mathbb{R}_+^N}] \text{ and } \overline{u}_{\infty,0}^{\mathbb{R}_+^N} = T_{\ell'}^1[\overline{u}_{\infty,0}^{\mathbb{R}_+^N}].$$

Thus $\underline{u}_{\infty,0}^{\mathbb{R}_+^N}$ and $\overline{u}_{\infty,0}^{\mathbb{R}_+^N}$ are self-similar solutions of (1.1) in \mathbb{R}_+^N vanishing on $\partial\mathbb{R}_+^N \setminus \{0\}$ and continuous in $\overline{\mathbb{R}_+^N} \setminus \{0\}$. Therefore they coincide with $u_{\infty,0}^{\mathbb{R}_+^N}$.

Finally, since

$$(4.45) \quad u_{\infty,0}^{B^\ell} \leq T_\ell^1[u_{\infty,0}^\Omega] \leq u_{\infty,0}^{G^\ell} \quad \forall 0 < \ell \leq 1,$$

by letting $\ell \rightarrow 0$ we obtain (4.39).

Case 2: H satisfies (1.8) with $p = \frac{q}{2-q}$. The proof is similar to the one in case 1.

Case 3: H satisfies (1.8) with $p > \frac{q}{2-q}$. For any $k > 0$ and $\ell > 0$, $T_\ell^2[u_{k,0}^\Omega]$ is a solution of (4.31) with boundary trace $\ell^{\beta_2+1-N}k\delta_0$. For $k > 0$, denote by Y_k^Ω the unique positive solution of

$$(4.46) \quad -\Delta Y + Y^p = 0 \quad \text{in } \Omega$$

with boundary trace $k\delta_0$ and by Y_∞^Ω the unique solution of (4.46) with strong singularity at the origin.

Since $0 < \ell < 1$ and $p > \frac{q}{2-q}$, by the comparison principle, we get

$$u_{\ell^{\beta_2+1-N}k,0}^{\Omega^\ell} \leq T_\ell^2[u_{k,0}^\Omega] \leq T_\ell^2[Y_k^\Omega] = Y_{\ell^{\beta_2+1-N}k}^{\Omega^\ell} \quad \text{in } \Omega^\ell.$$

By letting $k \rightarrow \infty$, we obtain

$$u_{\infty,0}^{\Omega^\ell} \leq T_\ell^2[u_{\infty,0}^\Omega] \leq Y_\infty^{\Omega^\ell} \quad \text{in } \Omega^\ell.$$

By Proposition 4.9 and [20], letting $\ell \rightarrow 0$ we deduce that

$$\lim_{\ell \rightarrow 0} \ell^{\beta_2} u_{\infty,0}^\Omega(\ell x) = Y_\infty^{\mathbb{R}_+^N}(x) = |x|^{\beta_2} \omega_3(x/|x|)$$

which implies (4.40) with $i = 3$.

Case 4: H satisfies (1.8) with $p < \frac{q}{2-q}$. Denote by $Z_\infty^{\mathbb{R}_+^N}$ the positive solution of

$$(4.47) \quad -\Delta Z + |\nabla Z|^q = 0 \quad \text{in } \mathbb{R}_+^N$$

with strong singular at the origin. By proceeding as in case 3 and results in [22], we derive

$$\lim_{\ell \rightarrow 0} \ell^{\beta_2} u_{\infty,0}^\Omega(\ell x) = Z_\infty^{\mathbb{R}_+^N}(x) = |x|^{\beta_2} \omega_4(x/|x|).$$

Thus (4.40) with $i = 4$ follows. \square

We next construct the maximal strongly singular solution.

Proposition 4.13. (i) Assume either H satisfies (1.7) with $0 < N(p+q-1) < p+1$ then there exists a maximal element $U_{\infty,0}^\Omega$ of \mathcal{U}_0^Ω . In addition, if $p \geq 1$ then

$$(4.48) \quad r^{\beta_1} U_{\infty,0}^\Omega(x) \rightarrow \omega_1(\sigma) \quad \text{as } r \rightarrow 0, \quad x \in \Omega, \quad \sigma \in S_+^{N-1}$$

locally uniformly on S_+^{N-1} .

(ii) If H satisfies (1.8) with $m_{p,q} < p_c$ then there exists a maximal element $U_{\infty,0}^\Omega$ of \mathcal{U}_0^Ω and

$$(4.49) \quad r^{\beta_2} U_{\infty,0}^\Omega(x) \rightarrow \omega_i(\sigma) \quad \text{as } r \rightarrow 0, \quad x \in \Omega, \quad \sigma \in S_+^{N-1}$$

locally uniformly on S_+^{N-1} where $i \geq 2$ is as in (4.38).

Proof. Case 1: H satisfies (1.7).

Step 1: Construction maximal solution. Since $0 < N(p+q-1) < p+1$, there exists a radial solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$ of the form

$$(4.50) \quad U_1^\dagger(x) = \Lambda_1^\dagger |x|^{-\beta_1} \quad \text{with} \quad \Lambda_1^\dagger = \left(\frac{\beta_1 + 2 - N}{\beta_1^{q-1}} \right)^{\frac{1}{p+q-1}}.$$

Therefore, $U_1^*(x) = \Lambda_1^* |x|^{-\beta_1}$ with $\Lambda_1^* = \max\{\Lambda_1^\dagger, \Lambda_1\}$ (here Λ_1 is the constant in (4.1)) is a supersolution of (1.1) in $\mathbb{R}^N \setminus \{0\}$ and dominates u in Ω . Let $\{\psi_{\epsilon,n}\}$ with $0 < \epsilon < \max\{|z| : z \in \Omega\}$ be a decreasing smooth sequence on $(\partial\Omega \setminus B_\epsilon(0)) \cup (\Omega \cap \partial B_\epsilon(0))$ such that

$$0 \leq \psi_{\epsilon,n} \leq \Lambda_1^* \epsilon^{-\beta_1}, \quad \psi_{\epsilon,n}(x) = \Lambda_1^* \epsilon^{-\beta_1} \quad \text{if } x \in \Omega \cap \partial B_\epsilon(0)$$

$$\psi_{\epsilon,n}(x) = 0 \quad \text{if } x \in \partial\Omega \setminus B_\epsilon(0) \text{ and } \text{dist}(x, \partial B_\epsilon(0)) > \frac{1}{n}.$$

Let $u_{\epsilon,n}^\Omega$ be the solution of

$$(4.51) \quad \begin{cases} -\Delta u + H \circ u = 0 & \text{in } \Omega \setminus B_\epsilon(0) \\ u = \psi_{\epsilon,n} & \text{on } (\partial\Omega \setminus B_\epsilon(0)) \cup (\Omega \cap \partial B_\epsilon(0)) \end{cases}$$

By the comparison principle, $u_{\epsilon,n}^\Omega \leq U_1^*$ in $\Omega \setminus B_\epsilon(0)$. Owing to Corollary 3.1, $\{u_{\epsilon,n}^\Omega\}$ converges to the solution u_ϵ^Ω of

$$(4.52) \quad \begin{cases} -\Delta u_\epsilon + H \circ u_\epsilon = 0 & \text{in } \Omega \setminus B_\epsilon(0) \\ u_\epsilon = 0 & \text{on } \partial\Omega \setminus B_\epsilon(0) \\ u_\epsilon = \Lambda_1^* \epsilon^{-\beta_1} & \text{on } \Omega \cap \partial B_\epsilon(0). \end{cases}$$

Consequently, $u_\epsilon^\Omega \leq U_1^*$. If $\epsilon' < \epsilon$, for n large enough, $u_{\epsilon',n}^\Omega \leq u_{\epsilon,n}^\Omega$, therefore

$$(4.53) \quad u_{\epsilon'}^\Omega \leq u_\epsilon^\Omega \leq U_1^* \quad \text{in } \Omega.$$

Letting ϵ to zero, $\{u_\epsilon^\Omega\}$ decreases and converges to some $U_{\infty,0}^\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$. Therefore $U_{\infty,0}^\Omega \in \mathcal{U}_0^\Omega$. Moreover, there holds

$$(4.54) \quad u_{\infty,0}^\Omega \leq U_{\infty,0}^\Omega \leq U_1^*(x).$$

If $u \in \mathcal{U}_0^\Omega$ then $u \leq u_{\epsilon,n}^\Omega$. Consequently, $u \leq U_{\infty,0}^\Omega$. Therefore $U_{\infty,0}^\Omega$ is the maximal element of \mathcal{U}_0^Ω .

Step 2: Proof of (4.48). Assume $p \geq 1$. From the fact that

$$(4.55) \quad T_\ell^1[U_1^*] = U_1^* \quad \forall \ell > 0,$$

and Theorem 4.11, we deduce $U_{\infty,0}^{\mathbb{R}_+^N} \equiv u_{\infty,0}^{\mathbb{R}_+^N}$.

Next, let B and B' are two open balls tangent to $\partial\Omega$ at 0 such that $B \subset \Omega \subset G := (B')^c$. Note that

$$T_\ell^1[u_\epsilon^B] = u_{\frac{\epsilon}{\ell}}^{B^\ell} \quad \text{and} \quad T_\ell^1[u_\epsilon^G] = u_{\frac{\epsilon}{\ell}}^{G^\ell} \quad \forall \ell, \epsilon > 0$$

where u_ϵ^G is the solution of (4.52) in $G \setminus B_\epsilon(0)$. By letting $\epsilon \rightarrow 0$ we deduce that

$$(4.56) \quad T_\ell^1[U_{\infty,0}^B] = U_{\infty,0}^{B^\ell} \quad \text{and} \quad T_\ell^1[U_{\infty,0}^G] = U_{\infty,0}^{G^\ell}.$$

Notice that

$$(4.57) \quad U_{\infty,0}^{B^{\ell'}} \leq U_{\infty,0}^{B^\ell} \leq U_{\infty,0}^{\mathbb{R}_+^N} \leq U_{\infty,0}^{G^\ell} \leq U_{\infty,0}^{G^{\ell''}} \quad \forall 0 < \ell \leq \ell', \ell'' \leq 1$$

and

$$(4.58) \quad U_{\infty,0}^{B^{\ell'}} \leq U_{\infty,0}^{B^\ell} \leq T_\ell^1[U_{\infty,0}^\Omega] \leq U_{\infty,0}^{G^\ell} \leq U_{\infty,0}^{G^{\ell''}} \quad \forall 0 < \ell \leq \ell', \ell'' \leq 1.$$

Hence $U_{\infty,0}^{B^\ell} \uparrow \underline{U}_{\infty,0}^{\mathbb{R}_+^N} \leq U_{\infty,0}^{\mathbb{R}_+^N}$ and $U_{\infty,0}^{G^\ell} \downarrow \overline{U}_{\infty,0}^{\mathbb{R}_+^N} \geq U_{\infty,0}^{\mathbb{R}_+^N}$ as $\ell \rightarrow 0$ where $\underline{U}_{\infty,0}^{\mathbb{R}_+^N}$ and $\overline{U}_{\infty,0}^{\mathbb{R}_+^N}$ are positive solutions of (1.1) in \mathbb{R}_+^N which vanish on $\partial\mathbb{R}_+^N \setminus \{0\}$ and endow the same scaling invariance under T_ℓ^1 . Therefore they coincide with $u_{\infty,0}^{\mathbb{R}_+^N}$. Letting $\ell \rightarrow 0$ in (4.58) implies (4.48).

Case 2: H satisfies (1.8) with $p = \frac{q}{2-q}$. Since in this case, (1.1) admits a similarity transformation T_ℓ^2 , the proof is similar to the one in case 1.

Case 3: H satisfies (1.8) with $p > \frac{q}{2-q}$. In this case, (1.1) admits no similarity transformation and there is no radial solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$. We can instead employ a radial supersolution of the form

$$(4.59) \quad U_3^*(x) = \Lambda_3^* |x|^{-\beta_2} \quad \text{with} \quad \Lambda_3^* = \beta_2(\beta_2 + 2 - N)^{\frac{1}{p-1}}$$

and then we proceed to construct the maximal solution as in case 1. For $\epsilon > 0$, let u_ϵ^Ω be the solution of (4.52). Since $u_{\frac{\epsilon}{\ell}}^{\Omega^\ell} \leq T_\ell^2[u_\epsilon^\Omega]$, by letting $\epsilon \rightarrow 0$ we obtain $U_{\infty,0}^{\Omega^\ell} \leq T_\ell^2[U_{\infty,0}^\Omega]$. It follows that

$$u_{\infty,0}^{\Omega^\ell} \leq U_{\infty,0}^{\Omega^\ell} \leq T_\ell^2[U_{\infty,0}^\Omega] \leq T_\ell^2[Y_\infty^\Omega] = Y_\infty^{\Omega^\ell}$$

where Y_∞^Ω is the unique strongly singular solution of (4.46). Due to Proposition 4.9 and the uniqueness, we deduce

$$\lim_{\ell \rightarrow 0} T_\ell^2[U_{\infty,0}^\Omega] = Y_\infty^{\mathbb{R}_+^N},$$

which, together with the fact $Y_\infty^{\mathbb{R}_+^N}(x) = |x|^{-\beta_2} \omega_3(x/|x|)$, implies (4.49).

Case 4: H satisfies (1.8) with $p < \frac{q}{2-q}$. The proof is similar to the one in case 3. \square

Proposition 4.12 and Proposition 4.13 show that the minimal solution $u_{\infty,0}^\Omega$ behaves like the maximal solution $U_{\infty,0}^\Omega$ near the origin, which enables us to prove the following result.

Theorem 4.14. *Assume either H satisfies (1.7) with $N(p+q-1) < p+1$ and $p \geq 1$ or H satisfies (1.8) with $m_{p,q} < p_c$. Then $U_{\infty,0}^\Omega = u_{\infty,0}^\Omega$.*

Proof. **Case 1:** H satisfies (1.7) with $p \geq 1$.

We represent $\partial\Omega$ near 0 as the graph of a C^2 function ϕ defined in $\mathbb{R}^{N-1} \cap B_R$ and such that $\phi(0) = 0$, $\nabla\phi(0) = 0$ and

$$\partial\Omega \cap B_R = \{x = (x', x_N) : x' \in \mathbb{R}^{N-1} \cap B_R, x_N = \phi(x')\}.$$

We introduce the new variable $y = \Phi(x)$ with $y' = x'$ and $y_N = x_N - \phi(x')$, with corresponding spherical coordinates in \mathbb{R}^N , $(r, \sigma) = (|y|, \frac{y}{|y|})$.

Let u is a positive solution of (1.1) in Ω vanishing on $\partial\Omega \setminus \{0\}$. We set $u(x) = r^{-\beta_1} v(t, \sigma)$ with $t = -\ln r \geq 0$, then a technical computation shows that v satisfies with $\mathbf{n} = \frac{y}{|y|}$

$$\begin{aligned} (4.60) \quad & (1 + \epsilon_1^1) v_{tt} + (2\beta_1 + 2 - N + \epsilon_2^1) v_t + (\beta_1(\beta_1 + 2 - N) + \epsilon_3^1) v + \Delta' v \\ & + \langle \nabla' v, \vec{\epsilon}_4^1 \rangle + \langle \nabla' v_t, \vec{\epsilon}_5^1 \rangle + \langle \nabla' \langle \nabla' v, \mathbf{e}_N \rangle, \vec{\epsilon}_6^1 \rangle \\ & - v^p |(-\beta_1 v + v_t) \mathbf{n} + \nabla' v + \langle (-\beta_1 v + v_t) \mathbf{n} + \nabla' v, \mathbf{e}_N \rangle \vec{\epsilon}_7^1|^q = 0, \end{aligned}$$

on $Q_R := [-\ln R, \infty) \times S_+^{N-1}$ where ϵ_j^1 have the following properties

- ϵ_j^1 are uniformly continuous functions of t and $\sigma \in S^{N-1}$ for $j = 1, \dots, 7$,
- ϵ_j^1 are C^1 functions for $j = 1, 5, 6, 7$,
- $|\epsilon_j^1(t, \cdot)| \leq c_{12} e^{-t}$ for $j = 1, \dots, 7$ and $|\epsilon_{jt}^1(t, \cdot)| + |\nabla' \epsilon_j^1| \leq c_{12} e^{-t}$ for $j = 1, 5, 6, 7$.

Moreover v vanishes on $[-\ln R, \infty) \times \partial S_+^{N-1}$. By [8, Theorem 4.7], there exist constants $c_{13} > 0$ and $T > \ln R$ such that

$$(4.61) \quad \|v(t, \cdot)\|_{C^{2,\gamma}(\overline{S_+^{N-1}})} + \|v_t(t, \cdot)\|_{C^{1,\gamma}(\overline{S_+^{N-1}})} + \|v_{tt}(t, \cdot)\|_{C^{0,\gamma}(\overline{S_+^{N-1}})} \leq c_{13}$$

for $\gamma \in (0, 1)$ and $t \geq T + 1$. Moreover

$$\lim_{t \rightarrow \infty} \int_{S_+^{N-1}} (v_t^2 + v_{tt}^2 + |\nabla' v_t|^2) dS(\sigma) = 0.$$

Consequently, the ω -limit set of v

$$\Gamma^+(v) = \cap_{\tau \geq 0} \overline{\cup_{t \geq \tau} v(t, \cdot)}^{C^2(S_+^{N-1})}$$

is a non-empty, connected and compact subset of the set of \mathcal{E}_1 . By the uniqueness of (4.37), $\Gamma^+(v) = \mathcal{E}_1 = \{\omega_1\}$. Hence $\lim_{t \rightarrow \infty} v(t, \cdot) = \omega_1$ in $C^2(\overline{S_+^{N-1}})$.

By taking $u = u_{\infty,0}^\Omega$ and $u = U_{\infty,0}^\Omega$ we obtain

$$(4.62) \quad \lim_{\Omega \ni x \rightarrow 0} \frac{u_{\infty,0}^\Omega(x)}{U_{\infty,0}^\Omega(x)} = 1.$$

For any $\varepsilon > 0$, by the comparison principle, $(1 + \varepsilon)u_{\infty,0}^\Omega \geq U_{\infty,0}^\Omega$ in $\Omega \setminus B_\varepsilon$. Letting $\varepsilon \rightarrow 0$ yields $u_{\infty,0}^\Omega \geq U_{\infty,0}^\Omega$ in Ω and thus $u_{\infty,0}^\Omega = U_{\infty,0}^\Omega$ in Ω .

Case 2: H satisfies (1.8) with $p = \frac{q}{2-q}$. The desired result is obtained by a similar argument.

Case 3: H satisfies (1.8) with $p > \frac{q}{2-q}$. In this case, we use the transformation $t = -\ln r$ for $t \geq 0$ and $\tilde{u}(r, \sigma) = r^{-\beta_2} v(t, \sigma)$ and obtain the following equation instead of (4.60)

$$(4.63) \quad \begin{aligned} & (1 + \epsilon_1^3) v_{tt} + (2\beta_2 + 2 - N + \epsilon_2^3) v_t + (\beta_2(\beta_2 + 2 - N) + \epsilon_3^3) v + \Delta' v \\ & + \langle \nabla' v, \vec{\epsilon}_4^3 \rangle + \langle \nabla' v_t, \vec{\epsilon}_5^3 \rangle + \langle \nabla' \langle \nabla' v, \mathbf{e}_N \rangle, \vec{\epsilon}_6^3 \rangle - v^p \\ & - e^{-\frac{p(2-q)-q}{p-1}t} |(-\beta_1 v + v_t) \mathbf{n} + \nabla' v + \langle (-\beta_1 v + v_t) \mathbf{n} + \nabla' v, \mathbf{e}_N \rangle \vec{\epsilon}_7^3|^q = 0 \end{aligned}$$

where ϵ_j^3 has the same properties as ϵ_j^1 ($j = \overline{1, 7}$). Notice that

$$\lim_{t \rightarrow \infty} e^{-\frac{p(2-q)-q}{p-1}t} = 0$$

since $p > \frac{q}{2-q}$. By proceeding as in the Case 1, we deduce that $u_{\infty,0}^\Omega = U_{\infty,0}^\Omega$ in Ω .

Case 4: H satisfies (1.8) with $p < \frac{q}{2-q}$. Using a similar argument as in Case 3, we obtain $u_{\infty,0}^\Omega = U_{\infty,0}^\Omega$ in Ω . \square

Proof of Theorem E. Statement (i) follows from Theorem 4.14, while statement (ii) follows from Proposition 4.12. \square

5. DIRICHLET PROBLEM WITH UNBOUNDED MEASURE DATA

Throughout this subsection we assume that H satisfies (1.8).

Proposition 5.1. *Assume H satisfies (1.8) with $p > 1$, $1 < q < 2$ and K is a compact subset of $\partial\Omega$. Then there exists $C > 0$ depending on N , p , q and the C^2 characteristic of Ω such that for any positive solution $u \in C(\overline{\Omega} \setminus K) \cap C^2(\Omega)$ of (1.1) vanishing on $\partial\Omega \setminus K$, there holds*

$$(5.1) \quad u(x) \leq C\rho(x)\rho_K(x)^{-\beta_2-1} \quad \forall x \in \Omega$$

where $\rho_K(x) = \text{dist}(x, K)$.

Proof. Since u is a positive solution of (1.1), it is a subsolution of

$$-\Delta v + v^p = 0$$

in Ω . By [21, Proposition 3.4.4], there exists a constant $C_1 > 0$ depending on N , p and the C^2 characteristic of Ω such that

$$(5.2) \quad u(x) \leq C_1\rho(x)\rho_K(x)^{-\frac{p+1}{p-1}} \quad \forall x \in \Omega.$$

Next put $\mathcal{C}_K = \{x \in \Omega : \rho(x) > \frac{1}{4}\rho_K(x)\}$. Since u is a positive subsolution of

$$-\Delta v + |\nabla v|^q = 0$$

in Ω , by a similar argument as in the proof of [22, Proposition 3.5], we can show that there exist positive constants $\delta^* \in (0, \delta_0)$ and $C_2 > 0$ depending on N , q and Ω such that

$$(5.3) \quad u(x) \leq C_2\rho(x)\rho_K(x)^{-\frac{1}{q-1}}$$

for every $x \in \Omega_{\delta^*} \setminus \mathcal{C}_K$. This, along with (2.5), implies that (5.3) holds in Ω . By combining (5.2) and (5.3), we deduce (5.1). \square

Lemma 5.2. *Let u and v be two positive solutions of (1.1). Assume that $u \geq v$ in Ω . Then either $u \equiv v$ or $u > v$ in Ω .*

Proof. Put $w = u - v$ then $w \geq 0$ in Ω and w satisfies

$$-\Delta w + a(x) \cdot \nabla w + b(x)w = 0 \quad \text{in } \Omega$$

where

$$a(x) = \begin{cases} \frac{(|\nabla u|^q - |\nabla v|^p) \nabla(u-v)}{|\nabla(u-v)|^2} & \text{if } \nabla u \neq \nabla v \\ 0 & \text{if } \nabla u = \nabla v, \end{cases}$$

$$b(x) = \begin{cases} \frac{u^p - v^p}{u-v} & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases}$$

By Proposition 2.4, $|a(x)| \leq c_1\rho(x)^{-1}$ and $b(x) \leq c_2\rho(x)^{-2}$ in Ω where c_1 and c_2 depend on N , p , q and δ_0 .

Next suppose that there exists $x_0 \in \Omega$ such that $w(x_0) = 0$. Let $r > 0$ such that $B_{3r}(x_0) \subset \Omega$. By Harnack inequality [24, Theorem 5], there exists $c_3 = c_3(N, p, q, \delta_0, x_0, r)$ such that

$$\max_{B_\delta(x_0)} w \leq c_3 \min_{B_\delta(x_0)} w = 0.$$

Hence $w \equiv 0$ in $B_r(x_0)$. By standard connectedness argument and Harnack inequality, we deduce that $w \equiv 0$ in Ω . \square

For any $k > 0$ and $y \in \partial\Omega$, let $u_{k,y}$ be the unique solution of (1.1) with boundary trace $k\delta_y$ and $u_{\infty,y}$ be the unique solution of (1.1) with strong singularity at y .

Proof of Theorem G. *Step 1: Construction of minimal element of \mathcal{U}_K .* Denote by \mathcal{V}_K the family of all positive moderate solutions u of (1.1) such that $u = 0$ on $\partial\Omega \setminus K$. Set $u_K := \sup \mathcal{V}_K$.

By Corollary 3.11, if $u, v \in \mathcal{V}_K$ then there exists a solution $\tilde{u} \in \mathcal{V}_K$ such that $\max(u, v) < \tilde{u}$. This fact and Lemma 5.2 imply (by the same proof as in [21, Lemma 3.2.1]) that u_K is the limit of an increasing sequence of solutions in \mathcal{V}_K . Proposition 2.4 implies that u_K is a solution of the equation and vanishes on $\partial\Omega \setminus K$. Clearly $u_K \geq \sup\{u_{\infty,y} : y \in K\}$. Therefore $\mathcal{S}(u_K) = K$ and $u \in \mathcal{U}_K$.

Next we show that u_K is the minimal element of \mathcal{U}_K .

If $w \in \mathcal{U}_K$ then by Lemma 4.7

$$w \geq \sup\{u_{\infty,y} : y \in K\} = \sup\{u_{k,y} : k > 0, y \in K\}.$$

By Theorem B and Corollary 3.11, w dominates every solution of (1.1) whose boundary trace belongs to $\mathbb{D}(K)$ (= set of finite linear combination of Dirac measures supported on K). If $u \in \mathcal{V}_K$ then from Theorem 3.7 we obtain $\text{tr}(u) = \mu \in \mathfrak{M}^+(\partial\Omega)$ with $\text{supp } \mu \subset K$. Hence there exists a sequence $\{\mu_m\} \subset \mathbb{D}(K)$ converging weakly to μ . By stability and uniqueness result, the sequence $\{u_{\mu_m}\}$ converges to u in $L^1(\Omega)$. Since $u_{\mu_m} \leq w$ for every n , we deduce that $u \leq w$. Therefore $u_K \leq w$ and u_K is the minimal element of \mathcal{U}_K .

Step 2: Construction of maximal element of \mathcal{U}_K . Denote by \mathcal{W}_K the family of all positive solutions u of (1.1) such that $u = 0$ on $\partial\Omega \setminus K$. Put $U_K := \sup \mathcal{W}_K$. By the same argument as in Step 1, one shows that $U_K \in \mathcal{W}_K$. By Lemma 4.7, $U_K \geq \sup\{u_{\infty,y} : y \in K\}$, which implies $\mathcal{S}(U_K) = K$. Therefore U_K is the maximal element of \mathcal{U}_K .

Step 3: Proof of (1.21). Pick $y \in K$. We may assume y is the origin. By Proposition 4.12, for every $\gamma \in (0, 1)$, there exists $r = r(\gamma)$ and $c = c(N, p, q, \gamma)$ such that

$$(5.4) \quad u_{\infty,y}^\Omega(x) \geq c|x - y|^{-\beta_2} \quad \forall x \in C_{\gamma,r}(y) := \{x \in \Omega : \rho(x) \geq \gamma|x - y|\} \cap B_r(y).$$

Since $y \in K = \mathcal{S}(u_K)$ we have $u_K \geq u_{\infty,y}$. Therefore

$$(5.5) \quad u_K(x) \geq c|x - y|^{-\beta_2} \quad \forall x \in C_{\gamma,r}(y).$$

On the other hand, by Proposition 2.4, for every $x \in C_{\gamma,r}(y)$,

$$(5.6) \quad U_K(x) \leq \tilde{\Lambda}_2 \rho(x)^{-\beta_2} \leq \tilde{\Lambda}_2 \gamma^{-\beta_2} |x - y|^{-\beta_2}.$$

From (5.5) and (5.6) we deduce (1.21). \square

6. REMOVABILITY

In this section we deal with removable singularities in the case that H is supercritical.

Proposition 6.1. *Assume either H satisfies (1.7) with $N(p + q - 1) \geq p + 1$ or H satisfies (1.8) with $m_{p,q} \geq p_c$. If $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ is a nonnegative solution of (1.1) vanishing on $\partial\Omega \setminus \{0\}$ then u cannot be a strongly singular solution.*

Proof. We consider a sequence of functions $\zeta_n \in C^\infty(\mathbb{R}^N)$ such that $\zeta_n(x) = 0$ if $|x| \leq \frac{1}{n}$, $\zeta_n(x) = 1$ if $|x| \geq \frac{2}{n}$, $0 \leq \zeta_n \leq 1$ and $|\nabla \zeta_n| \leq c_{13}n$, $|\Delta \zeta_n| \leq c_{13}n^2$ where c_{13} is independent of n . We take $\xi \zeta_n$ as a test function (where ξ is the solution to (3.3)) and we obtain

$$(6.1) \quad \int_{\Omega} (u + (H \circ u)\xi) \zeta_n dx = \int_{\Omega} u (\xi \Delta \zeta_n + 2\nabla \xi \cdot \nabla \zeta_n) dx =: J + J'.$$

Set $\mathcal{O}_n = \Omega \cap \{x : \frac{1}{n} < |x| \leq \frac{2}{n}\}$, then $|\mathcal{O}_n| \leq c_{14}(N)n^{-N}$. On the one hand, since $\xi(x) \leq c_3 \rho(x) \leq c_3|x|$,

$$J \leq c_{15} \Lambda_i \int_{\mathcal{O}_n} n^{\beta_i+2} \xi dx \leq c_{16} n^{\beta_i+1-N}$$

where

$$(6.2) \quad i = \begin{cases} 1 & \text{if } H \text{ satisfies (1.7),} \\ 2 & \text{if } H \text{ satisfies (1.8).} \end{cases}$$

On the other hand,

$$(6.3) \quad J' \leq c_{17} \Lambda_i \int_{\mathcal{O}_n} n^{\beta_i+1} |\nabla \xi| dx \leq c_{18} n^{\beta_i+1-N}$$

where i is given in (6.2). By combining (6.1)-(6.3) and then by letting $n \rightarrow \infty$ we obtain

$$(6.4) \quad \int_{\Omega} (u + (H \circ u)\xi) dx < \infty.$$

By Theorem 3.7, the boundary trace of u is a finite measure. Since $u = 0$ on $\partial\Omega \setminus \{0\}$, the boundary trace of u is $k\delta_0$ for some $k \geq 0$. \square

Corollary 6.2. *Assume either H satisfies (1.7) with $N(p + q - 1) > p + 1$ or H satisfies (1.8) with $m_{p,q} > p_c$. If $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ is a nonnegative solution of (1.1) vanishing on $\partial\Omega \setminus \{0\}$ then $u \equiv 0$.*

Proof. Since $\beta_i + 1 - N < 0$, we deduce from (6.1)-(6.3) that

$$\int_{\Omega} (u + (H \circ u)\xi) dx = 0,$$

which implies $u \equiv 0$. \square

Theorem 6.3. *Assume either H satisfies (1.7) with $N(p + q - 1) = p + 1$ or H satisfies (1.8) with $m_{p,q} = p_c$. If $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ is a nonnegative solution of (1.1) vanishing on $\partial\Omega \setminus \{0\}$ then $u \equiv 0$.*

Proof. By Proposition 6.1, u admits a boundary trace $k\delta_0$, $k \geq 0$.

For $0 < \ell < 1$, we set

$$u_{\ell}(x) = T_{\ell}^1[u](x) = T_{\ell}^2[u](x) = \ell^{N-1}u(\ell x), \quad x \in \Omega^{\ell} = \ell^{-1}\Omega.$$

By the comparison principle, $u_{\ell} \leq kP^{\Omega^{\ell}}(., 0)$ in Ω^{ℓ} for every $\ell \in (0, 1)$. Due to Proposition 4.9, up to a subsequence, $\{u_{\ell}\}$ converges to a function \tilde{u} which is a solution of either (4.27) if H satisfies (1.7), or (4.28) if H satisfies (1.8) with $p = \frac{q}{2-q}$, or (4.29) if H satisfies (1.8) with $p > \frac{q}{2-q}$, or (4.30) if H satisfies (1.8) with $p < \frac{q}{2-q}$. Moreover, $\tilde{u} \leq kP^{\mathbb{R}_+^N}(., 0)$ in \mathbb{R}_+^N .

If H satisfies (1.8) with $p \neq \frac{q}{2-q}$ then since $m_{p,q} = p_c$, it follows from [17] and [22] that $\tilde{u} = 0$.

If H satisfies (1.7) or H satisfies (1.8) with $p = \frac{q}{2-q}$ then set

$$\mathcal{V} = \{v : v \text{ is a solution of (1.1) in } \mathbb{R}_+^N, \tilde{u} \leq v \leq kP^{\mathbb{R}_+^N}(., 0)\}$$

and put $\tilde{v} := \sup \mathcal{V}$.

Assertion: \tilde{v} is a solution of (4.34) in \mathbb{R}_+^N .

Indeed, let $\{Q_n\}$ be a sequence of C^2 bounded domains such that $\overline{Q_n} \subset Q_{n+1}$, $\bigcup_{n \in \mathbb{N}} Q_n = \mathbb{R}_+^N$ and $0 < \text{dist}(Q_n, \partial\mathbb{R}_+^N) < \frac{1}{n}$ for each $n \in \mathbb{N}$. Consider the problem

$$(6.5) \quad \begin{cases} -\Delta w + H \circ w = 0 & \text{in } Q_n \\ w = kP^{\mathbb{R}_+^N}(., 0) & \text{on } \partial Q_n. \end{cases}$$

Since \tilde{u} and $kP^{\mathbb{R}_+^N}(., 0)$ are respectively subsolution and supersolution of (6.5), there exists a solution w_n of the problem (6.5) satisfying $\tilde{u} \leq w_n \leq kP^{\mathbb{R}_+^N}(., 0)$ in Q_n . Hence, by the comparison principle $\tilde{u} \leq w_{n+1} \leq w_n \leq kP^{\mathbb{R}_+^N}(., 0)$ in Q_n for each $n \in \mathbb{N}$. Therefore, $\tilde{w} := \lim_{n \rightarrow \infty} w_n \leq kP^{\mathbb{R}_+^N}(., 0)$ in \mathbb{R}_+^N . By regularity results [10], we obtain (4.33) with v_{ℓ} replaced by w_n and Ω^{ℓ} replaced by Q_n . Thus \tilde{w} is a solution of (4.34). On the one hand, by the definition of \tilde{v} , $\tilde{w} \leq \tilde{v}$. On the other hand, $\tilde{v} \leq w_n$ in Q_n for every n , and consequently $\tilde{v} \leq \tilde{w}$ in \mathbb{R}_+^N . Thus $\tilde{v} = \tilde{w}$.

For every $\ell > 0$, we set $w_{\ell} = T_{\ell}^1[\tilde{v}] = T_{\ell}^2[\tilde{v}] = \ell^{N-1}\tilde{v}(\ell x)$ with $x \in \mathbb{R}_+^N$ then $w_{\ell} = \sup \mathcal{V} = \tilde{v}$ in \mathbb{R}_+^N for every $\ell > 0$. Hence \tilde{v} is self-similar, namely \tilde{v} can be

written under the separable form

$$\tilde{v}(r, \sigma) = r^{N-1} \omega_i(\sigma) \quad (r, \sigma) \in (0, \infty) \times S_+^{N-1}$$

where ω_i is the nonnegative solution of (4.37). It follows from Theorem 4.10 that $\omega_i \equiv 0$, hence $\tilde{v} \equiv 0$. Thus $\tilde{u} \equiv 0$.

Hence

$$(6.6) \quad \lim_{n \rightarrow \infty} (\sup\{|u_{\ell_n}(x)| + |\nabla u_{\ell_n}(x)| : x \in \Gamma_{R^{-1}, R} \cap \Omega^{\ell_n}\}) = 0.$$

Consequently,

$$\lim_{x \rightarrow 0} |x|^{N-1} u(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} |x|^N |\nabla u(x)| = 0.$$

Therefore, $\lim_{x \rightarrow 0} (|x|^N \rho(x)^{-1} u(x)) = 0$, namely $u = o(P^\Omega(\cdot, 0))$. By the comparison principle, $u \equiv 0$. \square

Proof of Theorem F. The proof follows immediately from Corollary 6.2 and Theorem 6.3. \square

We next deal with the case $q = 2$.

Theorem 6.4. *Assume $q = 2$. If $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ is a nonnegative solution of (1.1) vanishing on $\partial\Omega \setminus \{0\}$ then $u \equiv 0$.*

Proof. Put

$$v = \begin{cases} 1 - e^{-\frac{1}{p+1} u^{p+1}} & \text{if } H \text{ satisfies (1.7),} \\ 1 - e^{-u} & \text{if } H \text{ satisfies (1.8)} \end{cases}$$

then $v \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$, $0 \leq v \leq 1$ and v satisfies

$$(6.7) \quad -\Delta v \leq 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega \setminus \{0\}.$$

Let η_δ be the solution of

$$(6.8) \quad -\Delta \eta_\delta = 0 \quad \text{in } D_\delta, \quad \eta_\delta = v \quad \text{on } \partial D_\delta$$

then by the comparison principle $v \leq \eta_\delta \leq 1$ in D_δ . The sequence $\{\eta_\delta\}$ converges to an harmonic function $\eta^* \geq v$ as $\delta \rightarrow 0$. Since $0 \leq \eta^* \leq 1$ and $\eta^* = 0$ on $\partial\Omega \setminus \{0\}$, it follows that $\eta^* \equiv 0$. Hence $v \equiv 0$, so $u \equiv 0$. \square

APPENDIX A. UNIQUENESS RESULT IN SUBCRITICAL CASE BY PHUOC-TAI NGUYEN

In this section, we deal with the question of uniqueness for the problem (1.9). Let Ω be a C^2 bounded domain in \mathbb{R}^N . We assume that $H \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N)$ satisfies

$$(A.1) \quad |H(x, u, \xi) - H(x, u', \xi')| \leq A \rho(x)^\alpha (a(x) + |\xi|^{q-1} + |\xi'|^{q-1}) |\xi - \xi'|$$

for a.e. $x \in \Omega$, for every $u, u' \in \mathbb{R}$ and $\xi, \xi' \in \mathbb{R}^N$, where $A > 0$, $\alpha \in (-1, \frac{1}{N-1})$,

$q \in (1, q_{\alpha, c})$ with $q_{\alpha, c} := \frac{N+1+\alpha}{N}$ and $a \in L^{\frac{q}{q-1+\alpha}}(\Omega)$. As above, we use the notation $H \circ u$ to denote $H(x, u(x), \nabla u(x))$.

Solutions of (1.9) are always understood in the sense of Definition 1.1.

The uniqueness result is stated as follows:

Theorem A.1. *Assume H satisfies (A.1). For every $\mu \in \mathfrak{M}^+(\partial\Omega)$, (1.9) admits at most one solution.*

The proof of Theorem A.1 is an adaptation of the method in [23] and based upon the following lemma

Lemma A.2. *Let $f \in L^1_\rho(\Omega)$ and z be a positive solution of*

$$(A.2) \quad -\Delta z \leq f \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \partial\Omega.$$

Then for any $\gamma \in (0, \frac{N}{N-1})$ and $1 < q < \frac{N+\gamma}{N}$, there exists a constant $c_{19} = c_{19}(N, \Omega, \gamma)$ such that

$$\|\nabla z\|_{L^q_{\rho^\gamma}(\Omega)} \leq c \|f\|_{L^1_\rho(\Omega)}.$$

Proof. Let \tilde{z} is the unique solution to the problem

$$(A.3) \quad -\Delta \tilde{z} = f \quad \text{in } \Omega, \quad \tilde{z} = 0 \quad \text{on } \partial\Omega.$$

then there exists a positive constant $c_{20} = c_{20}(N, \Omega, \gamma)$ such that

$$(A.4) \quad \|\nabla \tilde{z}\|_{L^q_{\rho^\gamma}(\Omega)} \leq c_{20} \|f\|_{L^1_\rho(\Omega)}.$$

Denote $\eta = \tilde{z} - z$ then η satisfies

$$-\Delta \eta \geq 0 \quad \text{in } \Omega, \quad \eta = 0 \quad \text{on } \partial\Omega.$$

By the maximum principle, $\eta \geq 0$ in Ω . As η is a superharmonic function, by [21, Theorem 1.4.1], there exists a positive Radon measure $\tau \in \mathfrak{M}^+(\Omega)$ such that

$$(A.5) \quad -\Delta \eta = \tau \quad \text{in } \Omega, \quad \eta = 0 \quad \text{on } \partial\Omega.$$

Therefore, by [4], for every $q \in (1, \frac{N+\gamma}{N})$, $|\nabla \eta| \in L^q_{\rho^\gamma}(\Omega)$ and

$$(A.6) \quad \|\nabla \eta\|_{L^q_{\rho^\gamma}(\Omega)} \leq c_{21} \|\tau\|_{\mathfrak{M}_\rho(\Omega)}$$

where c_{21} is another positive constant depending on N , Ω and γ . Moreover, by using ξ (ξ is the solution of (3.3)) as a test function, we deduce from (A.5) that

$$\|\eta\|_{L^1(\Omega)} = \int_\Omega \xi d\tau.$$

From (A.4) and (A.6), $z = \tilde{z} - \eta \in L^q_{\rho^\gamma}(\Omega)$ and

$$(A.7) \quad \|\nabla z\|_{L^q_{\rho^\gamma}(\Omega)} \leq c_{22} (\|f\|_{L^1_\rho(\Omega)} + \|\tau\|_{\mathfrak{M}_\rho(\Omega)})$$

where $c_{22} = c_{22}(N, \gamma, \Omega)$. Since $z \geq 0$, it follows that

$$(A.8) \quad \|\tau\|_{\mathfrak{M}_\rho(\Omega)} \leq c_3 \int_\Omega \xi d\tau = c_3 \|\eta\|_{L^1(\Omega)} \leq c \|\tilde{z}\|_{L^1(\Omega)} \leq c_{23} \|f\|_{L^1_\rho(\Omega)}$$

where $c_{23} = c_{23}(N, \gamma, \Omega)$. From (A.7) and (A.8) we get the desired estimate. \square

We turn to the

Proof of Theorem A.1. Let u_1 and u_2 be two solutions of (1.9). Since $q < q_{\alpha,c}$, from Proposition 2.5, we deduce that $|\nabla u_i| \in L_{\rho^{1+\alpha}}^q(\Omega)$, $i = 1, 2$ and

$$\|\nabla u_i\|_{L_{\rho^{1+\alpha}}^q(\Omega)} \leq c_1(\|H \circ u_i\|_{L_{\rho^1}^1(\Omega)} + \|\mu\|_{\mathfrak{M}(\partial\Omega)}).$$

Let $\{\mu_n\}$ be a sequence of functions in $C^1(\partial\Omega)$ converging weakly to μ . For $k > 0$, denote by T_k the truncation function, i.e. $T_k(s) = \max(-k, \min(s, k))$. For very $n > 0$, denote by $u_{i,n}$, $i = 1, 2$ the solution of the problem

$$(A.9) \quad -\Delta u_{i,n} + T_n(H \circ u_i) = 0 \quad \text{in } \Omega, \quad u_{i,n} = \mu_n \quad \text{on } \partial\Omega.$$

By local regularity theory for elliptic equations (see, e.g., [16]), $u_{i,n} \rightarrow u_i$ in $C_{loc}^1(\Omega)$ for $i = 1, 2$. From (A.9) we obtain

$$(A.10) \quad \begin{cases} -\Delta(u_{1,n} - u_{2,n}) &= -T_n(H \circ u_1) + T_n(H \circ u_2) \quad \text{in } \Omega \\ u_{1,n} - u_{2,n} &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

We shall prove that $u_1 \leq u_2$. By contradiction, we assume that $M := \sup_{\Omega}(u_1 - u_2) \in (0, \infty]$. Let $0 < k < M$. From (A.10) and Kato's inequality [21], we get

$$(A.11) \quad -\Delta(u_{1,n} - u_{2,n} - k)_+ \leq (T_n(H \circ u_2) - T_n(H \circ u_1))\chi_{E_{n,k}}$$

where $E_{n,k} = \{x \in \Omega : u_{1,n} - u_{2,n} > k\}$. Applying Lemma A.2 with $\gamma = 1 + \alpha$ and Holder's inequality, thanks to (A.1), we get

$$(A.12) \quad \begin{aligned} & \int_{\Omega} |\nabla(u_{1,n} - u_{2,n} - k)_+|^q \rho^{1+\alpha} dx \\ & \leq c_{24} \left(\int_{\Omega} |(T_n(H \circ u_2) - T_n(H \circ u_1))\chi_{E_{n,k}}| \rho dx \right)^q \\ & \leq c_{24} A^q \left(\int_{F_{n,k}} (a(x) + |\nabla u_1|^{q-1} + |\nabla u_2|^{q-1}) |\nabla(u_1 - u_2)| \rho^{1+\alpha} dx \right)^q \\ & \leq c_{25} \left(\int_{F_{n,k}} (a(x)^{\frac{q}{q-1}} + |\nabla u_1|^q + |\nabla u_2|^q) \rho^{1+\alpha} dx \right)^{q-1} \int_{F_{n,k}} |\nabla(u_1 - u_2)|^q \rho^{1+\alpha} dx \end{aligned}$$

where $F_{n,k} = \{x \in \Omega : u_{1,n} - u_{2,n} > k, \nabla u_1 \neq \nabla u_2\}$. Since $u_{1,n} - u_{2,n} \rightarrow u_1 - u_2$ a.e. in Ω , $\chi_{F_{n,k}} \rightarrow \chi_{F_k}$ a.e. where $F_k = \{x \in \Omega : u_1 - u_2 > k, \nabla u_1 \neq \nabla u_2\}$. Hence, letting $n \rightarrow \infty$ in (A.12) implies

$$(A.13) \quad \int_{\Omega} |\nabla(u_1 - u_2 - k)_+|^q \rho^{1+\alpha} dx \leq c_{25} R_k \int_{F_k} |\nabla(u_1 - u_2 - k)_+|^q \rho^{1+\alpha} dx$$

where

$$R_k := \left(\int_{F_k} (a(x)^{\frac{q}{q-1}} + |\nabla u_1|^q + |\nabla u_2|^q) \rho^{1+\alpha} dx \right)^{q-1}.$$

Case 1: $M = \infty$. Then $\lim_{k \rightarrow \infty} |F_k| = 0$. Since $a \in L^{\frac{q}{q-1}}_{\rho^{1+\alpha}}(\Omega)$ and $|\nabla u_1|, |\nabla u_2| \in L^q_{\rho^{1+\alpha}}(\Omega)$, there exists k_0 large enough so that $R_{k_0} < (2c_{25})^{-1}$. Hence, (A.13) implies

$$(A.14) \quad \int_{\Omega} |\nabla(u_1 - u_2 - k_0)_+|^q \rho^{1+\alpha} dx = 0.$$

It follows that $\nabla(u_1 - u_2 - k_0)_+ = 0$ in Ω and hence $(u_1 - u_2 - k_0)_+ = c_0 \geq 0$. Therefore, $u_1 - u_2 \leq k_0 + c_0$ a.e. in Ω , which contradicts $\sup_{\Omega}(u_1 - u_2) = \infty$.

Case 2: $M < \infty$. Since $|\nabla(u_1 - u_2)| = 0$ a.e. on the set $\{x \in \Omega : (u_1 - u_2)(x) = M\}$, it follows that $\lim_{k \rightarrow M} |F_k| = 0$. By proceeding as in Step 1, we deduce that there exists $k_0 \in (0, M)$ and $c_0 \geq 0$ such that $(u_1 - u_2 - k_0)_+ = c_0$ a.e. in Ω . If $c_0 = 0$ then $u_1 - u_2 \leq k_0$, which contradicts $\sup_{\Omega}(u_1 - u_2) = M > k_0$. If $c_0 > 0$ then $u_1 - u_2 = k_0 + c_0$. Then $u_1 - u_2 = M$ a.e. in Ω , which contradicts the fact that u and u_2 have the same boundary trace μ .

Thus $u_1 \leq u_2$ and the uniqueness follows by permuting the roles of u_1 and u_2 . \square

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